Chapter 17

Multiple Integration

Functions can be integrated over regions in the plane and in space.

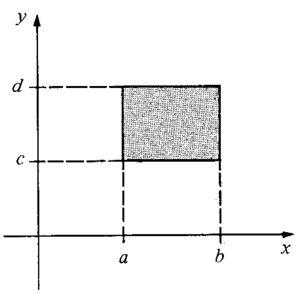
Double and triple integrals enable us to "sum" the values of real-valued functions of two or three variables; we evaluate them by integration with respect to one variable at a time, using the methods of one variable calculus. As in the previous chapters, we concentrate on the basic ideas and methods of calculation, leaving a few of the more theoretical points for a later course.

17.1 The Double Integral and Iterated Integral

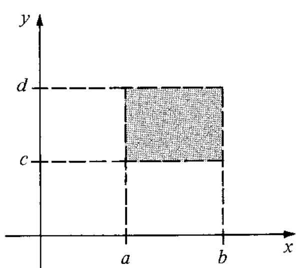
The double integral of a non-negative function over a region in the plane is equal to the volume under its graph.

The definite integral $\int_a^b f(x)\,dx$, defined in Chapter 4, represents a "sum" of the values of f at the (infinitely many) points of the interval $[a,b]$. To "sum" the values of a function $f(x, y)$ over the points of a region D in the plane, we will define the *double integral* $\iint_D f(x, y)\,dx\,dy$. We recommend a rapid review of Sections 4.1 to 4.5 as preparation for the present section.

Our development of double integrals will be similar to that of definite integrals in Chapter 4. We will give a formal definition first, but the actual calculation of double integrals will be done by reduction to repeated ordinary integrals as explained later in the section, rather than using the formal definition.



(a) The closed rectangle $a \leqslant x \leqslant b, c \leqslant y \leqslant d$



(b) The open rectangle $a < x < b, c < y < d$

Figure 17.1.1. Examples of closed and open rectangles.

The sets in the plane which will play the role of "intervals" in double integration are the rectangles (see Fig. 17.1.1). A *closed rectangle D* consists of all x and y such that $a \leqslant x \leqslant b$ and $c \leqslant y \leqslant d$; it is denoted by $[a,b] \times [c,d]$. The *interior* of D consists of all x and y such that $a < x < b$ and $c < y < d$; it is called an *open rectangle* and is denoted by $(a,b) \times (c,d)$. The *area* of D is the product $(b-a)(d-c)$. Note that the rectangles considered here have their sides parallel to the coordinate axes.

We say that a function $g(x, y)$ defined in $[a,b] \times [c,d]$ is a *step function* provided there are partitions $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ of $[a,b]$ and $c = s_0 < s_1 < s_2 < \cdots < s_m = d$ of $[c,d]$ such that, in each of the mn open rectangles $R_{ij} = (t_{i-1}, t_i) \times (s_{j-1}, s_j)$, $g(x, y)$ has a constant value k_{ij}. The graph of a step function is shown in Fig. 17.1.2.

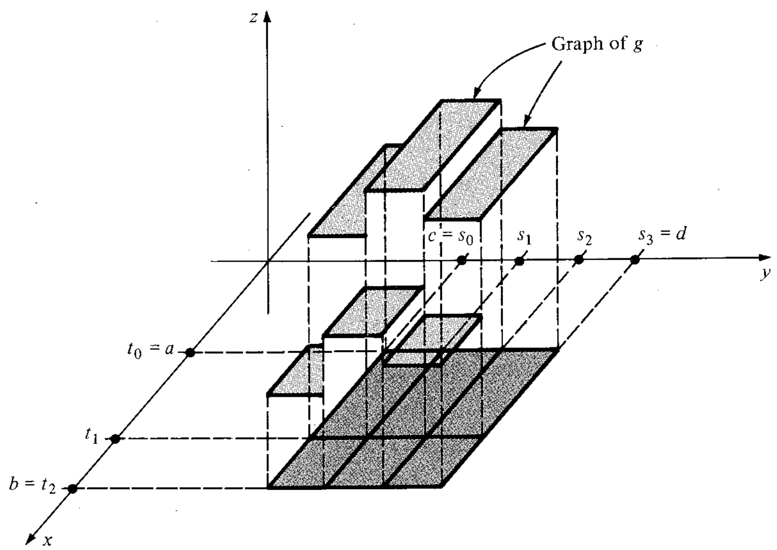


Figure 17.1.2. g is a step function since it is constant on each subrectangle.

By analogy with our definition for step functions of one variable, we define:

$$\iint_D g(x, y)\,dx\,dy = \sum_{i=1,j=1}^{n,m} (k_{ij})(\text{area } R_{ij}) = \sum_{i=1,j=1}^{n,m} (k_{ij})(\Delta t_i)(\Delta s_j),$$

where $\Delta t_i = t_i - t_{i-1}$ and $\Delta s_j = s_j - s_{j-1}$. The summation symbol means that we sum over all i and j, with i ranging from 1 to n and j from 1 to m; there are nm terms in the sum corresponding to the nm rectangles R_{ij}.

If $g(x, y) \geqslant 0$, the integral of g is exactly the volume under its graph. Indeed, the height of the box over the rectangle R_{ij} is k_{ij}, so the volume of the box is $k_{ij} \times \text{area}(R_{ij}) = k_{ij}\,\Delta t_i\,\Delta s_j$; the integral of g is the sum of these and so it is the total volume.

Example 1 Let g take values on the rectangles as shown in Fig. 17.1.3. Calculate the integral of g over the rectangle $D = [0, 5] \times [0, 3]$.

Solution The integral of g is the sum of the values of g times the areas of the rectangles:

$$\iint_D g(x, y)\,dx\,dy = -8 \times 2 + 2 \times 3 + 6 \times 2 + 3 \times 3 + 4 \times 2 - 1 \times 3$$

$$= 16. \blacktriangle$$

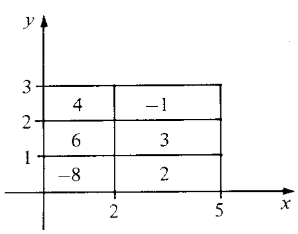


Figure 17.1.3. Find $\iint_D g(x, y)\,dx\,dy$ if g takes the values shown.

We proceed now to define the integral of a function over a closed rectangle in the same manner as we did in Section 4.3.

Upper and Lower Sums

The integral over D of a step function g such that $g(x, y) \leqslant f(x, y)$ on D is called a *lower sum* for f on D. The integral over D of a step function h such that $h(x, y) \geqslant f(x, y)$ on D is called an *upper sum* for f on D.

As in Section 4.3, every lower sum is less than or equal to every upper sum. The integral separates these sets of numbers.

The Double Integral

We say that f is *integrable* on D if there is a number S_0 such that every $S < S_0$ is a lower sum for f on D and every $S > S_0$ is an upper sum. The number S_0 is called the *integral* of f over D and is denoted by

$$\int\int_D f(x, y)\,dx\,dy.$$

Example 2 Let D be the rectangle $0 \leqslant x \leqslant 2$, $1 \leqslant y \leqslant 3$, and let $f(x, y) = x^2y$. Choose a step function $h(x, y) \geqslant f(x, y)$ to show that $\int\int_D f(x, y)\,dx\,dy \leqslant 25$.

Solution The constant function $h(x, y) = 12 \geqslant f(x, y)$ has integral $12 \times 4 = 48$, so we get only the crude estimate $\int\int_D f(x, y)\,dx\,dy \leqslant 48$. To get a better one, divide D into four pieces:

$$D_1 = [0,1] \times [1,2], \qquad D_3 = [1,2] \times [1,2],$$
$$D_2 = [0,1] \times [2,3], \qquad D_4 = [1,2] \times [2,3].$$

Let h be the step function given by taking the maximum value of f on each subrectangle (evaluated at the upper right-hand corner); that is,

$$h(x, y) = 2 \text{ on } D_1, 3 \text{ on } D_2, 8 \text{ on } D_3, \text{ and } 12 \text{ on } D_4.$$

The integral of h is $2 \times 1 + 3 \times 1 + 8 \times 1 + 12 \times 1 = 25$. Since $h \geqslant f$, we get

$$\int\int_D f(x, y)\,dx\,dy \leqslant 25. \ \blacktriangle$$

The basic properties of the double integral are similar to those of the ordinary integral:

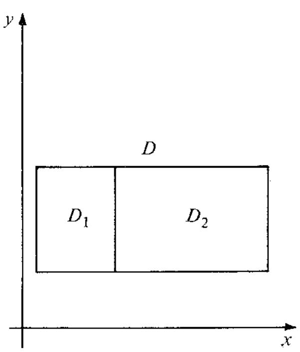


Figure 17.1.4. The rectangle D is divided into two smaller rectangles D_1 and D_2.

Properties of the Double Integral

1. Every continuous function is integrable.
2. If a rectangle D is divided by a line segment into two rectangles D_1 and D_2 (Fig. 17.1.4), and if $f(x, y)$ is integrable on D_1 and D_2, then f is integrable on D and
$$\int\int_D f(x, y)\,dx\,dy = \int\int_{D_1} f(x, y)\,dx\,dy + \int\int_{D_2} f(x, y)\,dx\,dy.$$
3. If f_1 and f_2 are integrable on D and if $f_1 \leqslant f_2$ on D, then
$$\int\int_D f_1(x, y)\,dx\,dy \leqslant \int\int_D f_2(x, y)\,dx\,dy.$$
4. If $f(x, y) = k$ on D,
$$\int\int_D f(x, y)\,dx\,dy = k(\text{area of } D).$$
5. $$\int\int_D [f_1(x, y) + f_2(x, y)]\,dx\,dy = \int\int_D f_1(x, y)\,dx\,dy + \int\int_D f_2(x, y)\,dx\,dy.$$
6. $$\int\int_D cf(x, y)\,dx\,dy = c\int\int_D f(x, y)\,dx\,dy.$$

We omit the proofs of these results since they are similar to the one-variable case. Choosing $k = 1$ in 4, note that $\iint_D dx\,dy =$ area of D.

We observed earlier that if $g(x, y) \geqslant 0$ and g is a step function, then the integral of g is the volume under its graph. If $f(x, y) \geqslant 0$ is any integrable function, then the volume under the graph of f lies between the volumes under the graphs of step functions $g \leqslant f$ and $h \geqslant f$; that is, between lower and upper sums. Since the integral has exactly this property, we conclude, as for functions of one variable, that *the integral of f over D equals the volume under the graph of f if $f \geqslant 0$* (see Fig. 17.1.5).

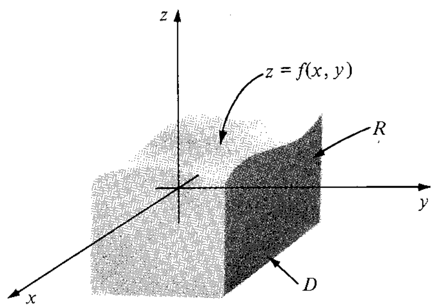


Figure 17.1.5. The volume of R, the region under the graph of f, equals $\iint_D f(x, y)\,dx\,dy$.

For the moment, we can evaluate integrals only approximately or by appealing to geometric formulas for volumes of special solids. Later in this section, we show how the fundamental theorem of calculus can be brought into play.

The double integral has other interpretations besides the volume of the region under the graph of the integrand. For example, suppose that a rectangular plate D has mass density $\rho(x, y)$ grams per square centimeter. Let us argue that $\iint_D \rho(x, y)\,dx\,dy$ is the mass of the plate. If ρ is constant, this is true since mass = density × area. Next, if ρ is a step function, then the integral of ρ over D is the mass of a plate with density ρ since the total mass is the sum of the masses of its parts. Now let ρ be arbitrary. If ρ_1 is a step function with $\rho_1 \leqslant \rho$, then $\iint_D \rho_1(x, y)\,dx\,dy \leqslant m$, where m is the mass of the plate with density ρ, since a lower density gives a smaller mass. Likewise, if ρ_2 is a step function with $\rho_2 \geqslant \rho$, then $m \leqslant \iint_D \rho_2(x, y)\,dx\,dy$. Thus the mass m lies between any pair of lower and upper sums for ρ, so it must equal the integral $\iint_D \rho(x, y)\,dx\,dy$.

Before establishing the fundamental result which will enable us to use one-variable techniques to evaluate double integrals, we must explain some notation. The *iterated integral*

$$\int_c^d \left[\int_a^b f(x, y)\,dx \right] dy$$

is evaluated, like most parenthesized expressions, from the inside out. One first holds y fixed and evaluates the integral $\int_a^b f(x, y)\,dx$ with respect to x; the result is a function of y which is then integrated from c to d.

The expression $\int_a^b [\int_c^d f(x, y)\,dy]\,dx$ is defined similarly; this time the integral with respect to y is evaluated first. Iterated integrals are often written without parentheses as

$$\int_c^d \int_a^b f(x, y)\,dx\,dy \quad \text{and} \quad \int_a^b \int_c^d f(x, y)\,dy\,dx.$$

Example 3 Evaluate $\int_0^2 \int_1^3 x^2y\,dy\,dx$.

Solution

$$\int_0^2 \int_1^3 x^2y\,dy\,dx = \int_0^2 \left(\int_1^3 x^2y\,dy \right) dx = \int_0^2 \left(\frac{x^2y^2}{2} \Big|_{y=1}^3 \right) dx$$

$$= \int_0^2 x^2\left(\frac{9}{2} - \frac{1}{2} \right) dx = 4\int_0^2 x^2\,dx = 4\frac{x^3}{3}\Big|_0^2 = \frac{32}{3}.$$

Notice that the second step in this calculation is essentially the inverse of a partial differentiation. ▲

We claim, and shall prove below, that the double integral equals the iterated integral. That is, for $D = [a,b] \times [c,d]$,

$$\int\int_D f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx.$$

To see why this might be so, let us suppose that $f(x,y) \geqslant 0$, so that the integral $\int\int_D f(x,y)\,dx\,dy$ represents the volume of the region R under the graph of f. If we take this volume and slice it by a plane parallel to the yz plane at a distance x from the origin, we get a two-dimensional region whose area is given by $A(x) = \int_c^d f(x,y)\,dy$ (see Fig. 17.1.6).

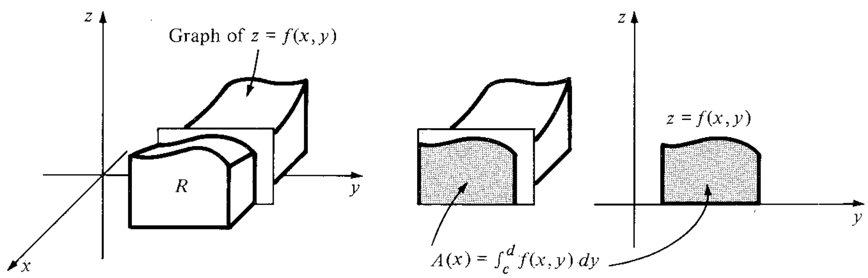


Figure 17.1.6. The area of the cross-section is the area under the graph of $z = f(x,y)$ from $y = c$ to $y = d$ (where x is fixed).

By Cavalieri's principle (Section 9.1), the total volume is the integral of the area function $A(x)$. Thus,

$$\int\int_D f(x,y)\,dx\,dy = \text{volume of } R = \int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx.$$

In the same way, if we use planes parallel to the xz plane, we get

$$\int\int_D f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy,$$

which is what we claimed. We see that Cavalieri's principle gives a geometric "proof" of the reduction to iterated integrals; in fact, it is more appropriate to take our proof below as a justification for Cavalieri's principle.

Theorem: Reduction to Iterated Integrals

Assume that $f(x,y)$ is integrable on the rectangle $D = [a,b] \times [c,d]$. Then any iterated integral which exists is equal to the double integral; that is,

$$\int\int_D f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx.$$

Proof of the Reduction to Iterated Integrals To prove this theorem, we first show it is true for step functions. Let g be a step function, with $g(x, y) = k_{ij}$ on $(t_{i-1}, t_i) \times (s_{j-1}, s_j)$, so that

$$\int\int_D g(x, y)\,dx\,dy = \sum_{i=1, j=1}^{n,m} k_{ij}\,\Delta t_i\,\Delta s_j .$$

If the summands $k_{ij}\,\Delta t_i\,\Delta s_j$ are laid out in a rectangular array, they may be added by first adding along rows and then adding up the subtotals, as follows:

$$\begin{array}{ccccl}
k_{11}\,\Delta t_1\,\Delta s_1 & k_{21}\,\Delta t_2\,\Delta s_1 & \cdots & k_{n1}\,\Delta t_n\,\Delta s_1 \longrightarrow & \left(\sum_{i=1}^{n} k_{i1}\,\Delta t_i\right)\Delta s_1 \\
k_{12}\,\Delta t_1\,\Delta s_2 & k_{22}\,\Delta t_2\,\Delta s_2 & \cdots & k_{n2}\,\Delta t_n\,\Delta s_2 \longrightarrow & \left(\sum_{i=1}^{n} k_{i2}\,\Delta t_i\right)\Delta s_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
k_{1m}\,\Delta t_1\,\Delta s_m & k_{2m}\,\Delta t_2\,\Delta s_m & \cdots & k_{nm}\,\Delta t_n\,\Delta s_m \longrightarrow & \left(\sum_{i=1}^{n} k_{im}\,\Delta t_i\right)\Delta s_m \\
\hline
 & & & & \sum_{j=1}^{m}\left(\sum_{i=1}^{n} k_{ij}\,\Delta t_i\right)\Delta s_j \; \downarrow
\end{array}$$

The coefficient of Δs_j in the sum over the jth row, $\sum_{i=1}^{n} k_{ij}\,\Delta t_i$, is equal to $\int_a^b g(x, y)\,dx$ for any y with $s_{j-1} < y < s_j$, since, for y fixed, $g(x, y)$ is a step function of x. Thus the integral $\int_a^b g(x, y)\,dx$ is a step function of y, and its integral with respect to y is the sum:

$$\int_c^d\left[\int_a^b g(x, y)\,dx\right]dy = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} k_{ij}\,\Delta t_i\right)\Delta s_j = \int\int_D g(x, y)\,dx\,dy.$$

Similarly, by summing first over columns and then over rows, we obtain

$$\int\int_D g(x, y)\,dx\,dy = \int_a^b\left[\int_c^d g(x, y)\,dy\right]dx.$$

The theorem is therefore true for step functions.

Now let f be integrable on $D = [a, b] \times [c, d]$ and assume that the iterated integral $\int_a^b[\int_c^d f(x, y)\,dy]\,dx$ exists. Denoting this integral by S_0, we will show that every lower sum for f on D is less than or equal to S_0, while every upper sum is greater than or equal to S_0, so S_0 must be the integral of f over D.

To carry out our program, let g be any step function such that

$$g(x, y) \leqslant f(x, y) \tag{1}$$

for all (x, y) in D. Integrating equation (1) with respect to y and using property 5 of the one-variable integral (see Section 4.5), we obtain

$$\int_c^d g(x, y)\,dy \leqslant \int_c^d f(x, y)\,dy \tag{2}$$

for all x in $[a, b]$. Integrating (2) with respect to x and applying property 5 once more gives

$$\int_a^b\left[\int_c^d g(x, y)\,dy\right]dx \leqslant \int_a^b\left[\int_c^d f(x, y)\,dy\right]dx. \tag{3}$$

Since g is a step function, it follows from the first part of this proof that the left-hand side of (3) is equal to the lower sum $\int\int_D g(x, y)\,dx\,dy$; the right-hand side of (3) is just S_0, so we have shown that every lower sum is less than or equal to S_0. The proof that every upper sum is greater than or equal to S_0 is similar, and so we are done. ■

Example 4 Let $f(x, y) = e^{2x+y}$. Evaluate the integral of f over $D = [0,1] \times [0,3]$.

Solution

$$\begin{aligned}\int\!\!\int_D f(x, y)\,dx\,dy &= \int_0^3\left(\int_0^1 e^{2x+y}\,dx\right)dy\\ &= \int_0^3\left(\frac{1}{2}e^{2x+y}\Big|_{x=0}^{1}\right)dy = \frac{1}{2}\int_0^3 (e^{2+y} - e^y)\,dy\\ &= \frac{1}{2}(e^2-1)\int_0^3 e^y\,dy = \frac{(e^2-1)(e^3-1)}{2} \approx 60.9693.\end{aligned}$$

(You should check that integrating with respect to y first gives the same answer.) ▲

Example 5 Evaluate $\int_1^3\int_0^2 x^2y\,dx\,dy$ and compare with Example 3.

Solution

$$\begin{aligned}\int_1^3\int_0^2 x^2y\,dx\,dy &= \int_1^3\left(\int_0^2 x^2y\,dx\right)dy = \int_1^3\left(\frac{x^3y}{3}\Big|_{x=0}^{2}\right)dy\\ &= \int_1^3 \frac{8}{3}y\,dy = \frac{4}{3}y^2\Big|_1^3 = \frac{4}{3}(3^2-1^2) = \frac{32}{3}.\end{aligned}$$

The answer is the same as that in Example 3, as predicted by the theorem above. (It is also consistent with Example 2.) ▲

Example 6 Compute $\int\!\!\int_D \sin(x+y)\,dx\,dy$, where $D = [0,\pi] \times [0, 2\pi]$.

Solution

$$\begin{aligned}\int\!\!\int_D \sin(x+y)\,dx\,dy &= \int_0^{2\pi}\left[\int_0^{\pi}\sin(x+y)\,dx\right]dy\\ &= \int_0^{2\pi}\left[-\cos(x+y)|_{x=0}^{\pi}\right]dy\\ &= \int_0^{2\pi}\left[\cos y - \cos(y+\pi)\right]dy\\ &= \left[\sin y - \sin(y+\pi)\right]\Big|_{y=0}^{2\pi} = 0.\ ▲\end{aligned}$$

Example 7 Find the volume under the graph of $f(x, y) = x^2 + y^2$ between the planes $x = 0$, $x = 3$, $y = -1$, and $y = 1$.

Solution The volume is

$$\begin{aligned}\int_{-1}^1\int_0^3 (x^2+y^2)\,dx\,dy &= \int_{-1}^1\left(\frac{x^3}{3} + y^2x\Big|_{x=0}^{3}\right)dy = \int_{-1}^1 (9+3y^2)\,dy\\ &= (9y+y^3)|_{-1}^{1} = 20.\ ▲\end{aligned}$$

Example 8 If D is a plate defined by $1 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 1$, and the mass density is $\rho(x, y) = ye^{xy}$ grams per square centimeter, find the mass of the plate.

Solution The total mass is

$$\int\!\!\int_D \rho(x, y)\,dx\,dy = \int_0^1\int_1^2 ye^{xy}\,dx\,dy = \int_0^1 \left(e^{xy}|_{x=1}^{2}\right)dy$$

$$= \int_0^1 (e^{2y} - e^y)\,dy = \left(\frac{e^{2y}}{2} - e^y\right)\Bigg|_{y=0}^{1}$$

$$= \frac{e^2}{2} - e + \frac{1}{2} \approx 1.4762 \text{ grams. } \blacktriangle$$

Supplement for Section 17.1: Solar Energy and Double Integrals

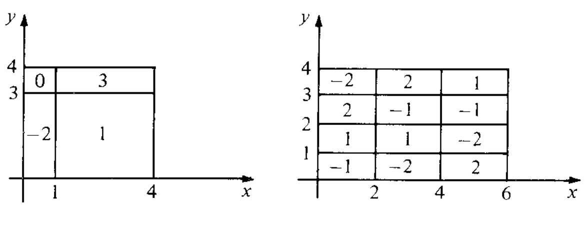
To illustrate the process of summation which is represented by the double integral, we may use an example connected with solar energy. The intensity of solar radiation is a "local" quantity, which may be measured at any point on the earth's surface. Since the solar intensity is really a rate of power input (see the Supplement to Section 9.5), we can measure it in units of watts per square meter.

If the solar intensity is uniform over a region, the total power received is equal to the intensity times the area of the region. In practice, the intensity is a function of position (in particular, it is a function of latitude); so we cannot just multiply a value by the area of the region. Instead, we must *integrate* the intensity over the region. Thus, the method of this section would allow us to find (at least in principle) the total power received by the state of Colorado, which is a rectangle in longitude–latitude coordinates. The problem for Utah is also tractable, since that state is composed of two rectangles, but what happens if we are interested in Michigan or Florida? For this problem, we need to integrate over regions which are not rectangles: the method for doing this is presented in the next section.

Exercises for Section 17.1

In Exercises 1 and 2, the function g takes values on the rectangles as indicated in Fig. 17.1.7. Calculate the integral of g.

1. For the rectangle in Fig. 17.1.7(a).
2. For the rectangle in Fig. 17.1.7(b).



(a) (b)

Figure 17.1.7. Find the integral of g.

3. (a) Let D be the rectangle determined by the inequalities $-1 \leqslant x \leqslant 1$ and $2 \leqslant y \leqslant 4$ and let $f(x, y) = x(1 + y)$. Find a step function g satisfying $g(x, y) \leqslant f(x, y)$ to show that $\iint_D f(x, y)\,dx\,dy \geqslant -9$.
(b) Sketch the graph of $f(x, y)$ over D. Use symmetry to argue that the value of the integral in part (a) must in fact be zero.

4. Let D be the rectangle $1 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 2$. Suppose that D is a plate with mass density $\rho(x, y) = 2xy^2 + \cos \pi y + 1$ (grams per square centimeter). Find step functions to show that the mass m (in grams) of the plate satisfies the inequalities $30 \leqslant m \leqslant 62$.

Evaluate the iterated integrals in Exercises 5–10.

5. $\int_0^3 \int_0^2 x^3 y\,dx\,dy$
6. $\int_0^2 \int_1^4 x^3 y\,dy\,dx$
7. $\int_0^2 \int_{-1}^1 (yx)^2\,dy\,dx$
8. $\int_{-1}^1 \int_0^2 (yx)^2\,dx\,dy$
9. $\int_{-1}^1 \int_0^1 y e^x\,dy\,dx$
10. $\int_{-1}^1 \int_0^3 y^5 e^{xy^3}\,dx\,dy$

Evaluate $\iint_D f(x, y)\,dx\,dy$ for the indicated functions and rectangles in Exercises 11–16.

11. $f(x, y) = (x + 2y)^2$; $D = [-1, 2] \times [0, 2]$.
12. $f(x, y) = y^3 \cos^2 x$; $D = [-\pi/2, \pi] \times [1, 2]$.
13. $f(x, y) = x^2 + 2xy - y\sqrt{x}$; $D = [0, 1] \times [-2, 2]$.
14. $f(x, y) = xy^3 e^{x^2 y^2}$; $D = [1, 3] \times [1, 2]$.
15. $f(x, y) = xy + x/(y + 1)$; $D = [1, 4] \times [1, 2]$.
16. $f(x, y) = y^5 e^{y^3 \cos x} \sin x$; $D = [0, 1] \times [-1, 0]$.
17. Evaluate $\int_2^4 \int_{-1}^1 x(1 + y)\,dx\,dy$ and compare with Exercise 3.
18. Evaluate $\int_0^2 \int_1^4 (2xy^2 + \cos \pi y + 1)\,dx\,dy$ and compare with Exercise 4.

Sketch and find the volume under the graph of f between the planes $x = a$, $x = b$, $y = c$, and $y = d$ in Exercises 19 and 20.

19. $f(x, y) = x^3 + y^2 + 2$; $a = -1$, $b = 1$, $c = 1$, $d = 3$.
20. $f(x, y) = 2x + 3y^2 + 2$; $a = 0$, $b = 3$, $c = -2$, $d = 1$.

21. The density at each point of a 1 centimeter square (i.e., each side has length 1 centimeter) microchip is $4 + r^2$ grams per square centimeter, where r is the distance in centimeters from the point to the center of the chip. What is the mass of the chip?
22. Do as in Exercise 21, but now let r be the distance to the lower left-hand corner of the plate.

★23. Prove that the sum of two step functions defined on the same rectangle is again a step function.

★24. Prove that if $f(x, y) \leqslant g(x, y)$ for all (x, y) in the rectangle $[a, b] \times [c, d]$, then

$$\int_a^b \int_c^d f(x, y)\, dy\, dx \leqslant \int_c^d \int_a^b g(x, y)\, dx\, dy.$$

★25. The state of Colorado occupies the region between 33° and 41° latitude and 102° and 108° longitude. A degree of latitude is about 110 kilometers and a degree of longitude is about 83 kilometers. The intensity of solar radiation at time t on day T at latitude l is (in suitable units; see the Supplement to Section 9.5)

$$I = \cos l \sqrt{1 - \sin^2\alpha \cos^2\left(\frac{2\pi T}{365}\right)} \cos\left(\frac{2\pi t}{24}\right) + \sin l \sin \alpha \cos\left(\frac{2\pi T}{365}\right).$$

(a) What is the integrated solar energy over Colorado at time t on day T?
(b) Suppose that the result of part (a) is integrated with respect to t from t_1 to t_2. What does the integral represent?

17.2 The Double Integral Over General Regions

Double integrals over general regions become iterated integrals with variable endpoints.

Many applications involve double integrals $\iint_D f(x, y)\, dx\, dy$ over regions D which are not rectangles. For instance, the volume of a hemisphere, the mass of an elliptical plate, or the total solar power received by the state of Texas can be expressed as such integrals. We shall find that such integrals can be evaluated by iterated integration in a form slightly more complicated than that used for rectangles.

To begin, we must *define* what we mean by $\iint_D f(x, y)\, dx\, dy$ when D is not a rectangle. We shall assume that D is contained in some rectangle D^*. Let f^* be the function on D^* defined by

$$f^*(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{if } (x, y) \notin D. \end{cases}$$

(See Fig. 17.2.1.)

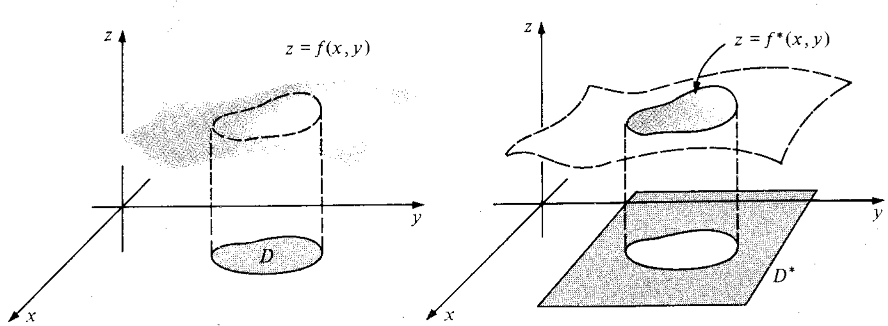


Figure 17.2.1. Given f and D, we construct f^* by setting $f^*(x, y)$ equal to zero outside D.

If D^{**} is another rectangle containing D, and f^{**} is the corresponding function defined as above, then $\iint_{D^*} f^*(x, y)\,dx\,dy = \iint_{D^{**}} f^{**}(x, y)\,dx\,dy$, since f^* and f^{**} are zero in the regions where D^* and D^{**} differ (see Fig. 17.2.2 and use the properties of the integral given in Section 17.1).

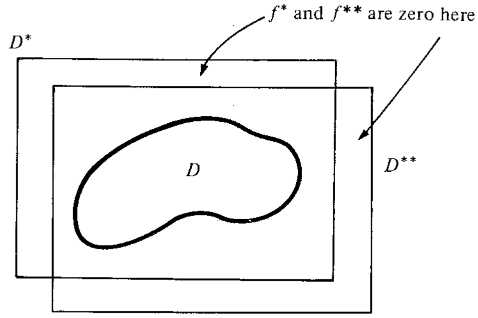


Figure 17.2.2. The choice of D^* does not matter.

We note that if $f(x, y) \geqslant 0$ on D, then both integrals above are equal to the volume of the region under the graph of f on D, i.e., the set of (x, y, z) such that $(x, y) \in D$ and $0 \leqslant z \leqslant f(x, y)$.

With these preliminaries, we can state the following definition.

The Double Integral Over a Region D

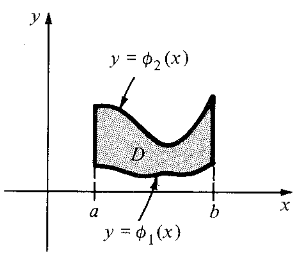
Extend f to a rectangle D^* containing D by letting f^* equal f on D and zero outside D. If f^* is integrable on D^*, then we say that f is *integrable* on D, and we define $\iint_D f(x, y)\,dx\,dy$ to be $\iint_{D^*} f^*(x, y)\,dx\,dy$. (By our preceding remarks, the choice of D^* does not affect the answer).

This definition serves the purpose of giving meaning to the double integral, but it is not very useful for computation. For this purpose, we need to choose D in a more specific way. We shall define two simple types of regions, which we will call *elementary regions*. Complicated regions can often be broken into elementary ones.

Suppose that we are given two continuous real-valued functions ϕ_1 and ϕ_2 on $[a, b]$ which satisfy $\phi_1(t) \leqslant \phi_2(t)$ for all t in $[a, b]$. Let D be the set of all points (x, y) such that

$$x \text{ is in } [a, b] \quad \text{and} \quad \phi_1(x) \leqslant y \leqslant \phi_2(x).$$

This region D is said to be of *type 1*. (See Fig. 17.2.3.) The curves and straight line segments that enclose the region constitute the *boundary* of D.



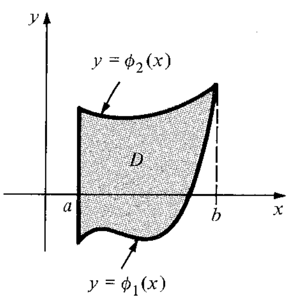


Figure 17.2.3. A region D is of type 1 if it is the region between the graphs of two functions, $y = \phi_1(x)$ and $y = \phi_2(x)$.

We say that a region D is of *type 2* if there are continuous functions ψ_1 and ψ_2 on $[c, d]$ such that D is the set of points (x, y) satisfying

$$y \text{ is in } [c, d] \quad \text{and} \quad \psi_1(y) \leqslant x \leqslant \psi_2(y),$$

where $\psi_1(t) \leqslant \psi_2(t)$ for t in $[c, d]$. (See Fig. 17.2.4.) Again the boundary of the region consists of the curves and line segments enclosing the region.

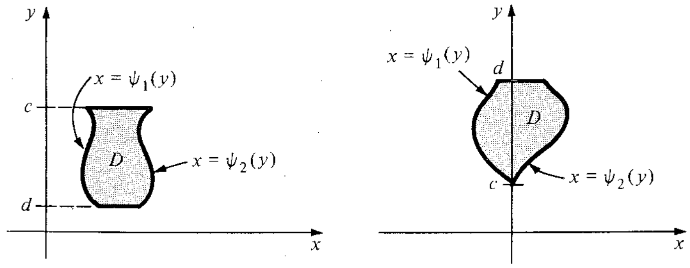
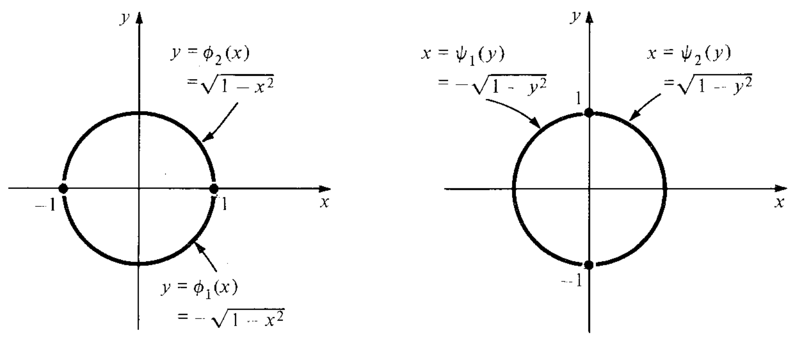


Figure 17.2.4. A region D is of type 2 if it is the region between the graphs of $x = \psi_1(y)$ and $x = \psi_2(y)$.

The following example shows that a given region may be of types 1 and 2 at the same time.

Example 1 Show that the region D defined by $x^2 + y^2 \leqslant 1$ (the unit disk) is a region of types 1 and 2.

Solution Descriptions of the disk, showing that it is of both types, are given in Fig. 17.2.5. ▲



(a) Type 1 region (b) Type 2 region

Figure 17.2.5. The unit disk as a type 1 region and a type 2 region.

We will use, without proof,[1] the following fact: If f is continuous on an elementary region D, then f is integrable on D. This fact, combined with the reduction to iterated integrals for rectangles, enables us to evaluate integrals over elementary regions by iterated integration. Indeed, if $D^* = [a,b] \times [c,d]$ is a rectangle containing D, then

$$\int\int_D f(x,y)\,dx\,dy = \int\int_{D^*} f^*(x,y)\,dx\,dy = \int_a^b \int_c^d f^*(x,y)\,dy\,dx \tag{1}$$

$$= \int_c^d \int_a^b f^*(x,y)\,dx\,dy, \tag{2}$$

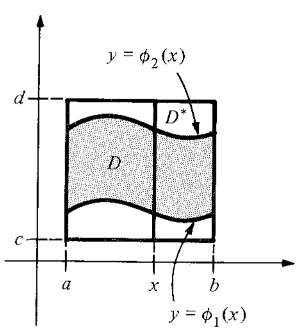


Figure 17.2.6. The integral of f^* over D^* equals that of f over D.

where f^* equals f in D and is zero outside D. Assume that D is a region of type 1 determined by functions ϕ_1 and ϕ_2 on $[a,b]$. Consider the iterated integral

$$\int_a^b \int_c^d f^*(x,y)\,dy\,dx$$

and, in particular, the inner integral $\int_c^d f^*(x,y)\,dy$ for some fixed x (Fig. 17.2.6). By definition, $f^*(x,y) = 0$ if $y < \phi_1(x)$ or $y > \phi_2(x)$, so

$$\int_c^d f^*(x,y)\,dy = \int_{\phi_1(x)}^{\phi_2(x)} f^*(x,y)\,dy = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy. \tag{3}$$

[1] For a proof, see an advanced calculus text, such as J. Marsden, *Elementary Classical Analysis*, Freeman (1974), Chapter 8.

Substituting (3) into (1) gives

$$\iint_D f(x, y)\,dx\,dy = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\,dy \right] dx.$$

A similar construction works for type 2 regions.

Double Integrals

If D is a region of type 1 (Fig. 17.2.3),

$$\iint_D f(x, y)\,dx\,dy = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\,dy \right] dx. \tag{4}$$

If D is of type 2 (Fig. 17.2.4),

$$\iint_D f(x, y)\,dx\,dy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\,dx \right] dy. \tag{5}$$

If D is of both types, either (4) or (5) is applicable.

If $f(x, y) \geqslant 0$ on D, we may understand the procedure of iterated integration as our old technique of finding volumes by slicing (Section 9.1). Suppose, for instance, that D is of type 1, determined by $\phi_1(x)$ and $\phi_2(x)$ on $[a, b]$. If we fix x and slice the volume under the graph of $f(x, y)$ on D by the plane which passes through the point $(x, 0, 0)$ and which is parallel to the yz plane, we obtain the region in the yz plane defined by the inequalities $\phi_1(x) \leqslant y \leqslant \phi_2(x)$ and $0 \leqslant z \leqslant f(x, y)$. The area $A(x)$ of this region is just $\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\,dy$. Now the double integral $\iint_D f(x, y)\,dx\,dy$, which is the volume of the entire solid, equals

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\,dy \right] dx,$$

the iterated integral. Thus we get (4).

If D is of type 2, then slicing by planes parallel to the xz plane produces the corresponding result (5). The reader should draw figures similar to Fig. 17.1.6 to accompany this discussion.

Example 2 Find $\iint_D (x + y)\,dx\,dy$, where D is the shaded region in Fig. 17.2.7.

Solution D is a region of type 1, with $[a, b] = [0, \frac{1}{2}]$, $\varphi_1(x) = 0$, and $\varphi_2(x) = x^2$. By formula (4) in the preceding box,

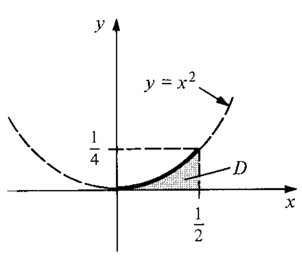


Figure 17.2.7. Find $\iint_D (x + y)\,dx\,dy$.

$$\iint_D (x + y)\,dx\,dy = \int_0^{1/2} \int_0^{x^2} (x + y)\,dy\,dx = \int_0^{1/2} \left[\left(xy + \frac{y^2}{2} \right) \Bigg|_{y=0}^{x^2} \right] dx$$

$$= \int_0^{1/2} \left(x^3 + \frac{x^4}{2} \right) dx = \left(\frac{x^4}{4} + \frac{x^5}{10} \right) \Bigg|_0^{1/2}$$

$$= \frac{1}{64} + \frac{1}{320} = \frac{3}{160}.$$

D is also a region of type 2, with $\psi_1(y) = \sqrt{y}$ and $\psi_2(y) = \frac{1}{2}$. We leave it to you to verify that the double integral calculated by formula (5) is also $\frac{3}{160}$. ▲

Example 3 Evaluate $\int_0^1 \int_{x^3}^{x^2} xy\,dy\,dx$. Sketch the region for the corresponding double integral.

Solution Here y ranges from x^3 to x^2, while x goes from 0 to 1. Hence the region is as shown in Fig. 17.2.8. The integral is

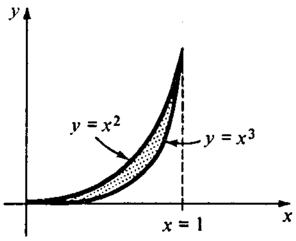


Figure 17.2.8. The region of integration for $\int_0^1 \int_{x^3}^{x^2} xy\,dy\,dx$.

$$\int_0^1 \left(\frac{xy^2}{2} \bigg|_{y=x^3}^{x^2} \right) dx = \int_0^1 \left(\frac{x^5}{2} - \frac{x^7}{2} \right) dx = \left(\frac{x^6}{12} - \frac{x^8}{16} \right) \bigg|_{x=0}^{1} = \frac{1}{12} - \frac{1}{16} = \frac{1}{48} \,. \blacktriangle$$

The next example shows that it sometimes saves labor to reverse the order of integration.

Example 4 Write $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy\,dx$ as an integral over a region. Sketch the region and show that it is of types 1 and 2. Reverse the order of integration and evaluate.

Solution The region D is shown in Fig. 17.2.9. D is a type 1 region with $\phi_1(x) = 0$, $\phi_2(x) = \sqrt{1-x^2}$ and a type 2 region with $\psi_1(y) = 0$, $\psi_2(y) = \sqrt{1-y^2}$. Thus

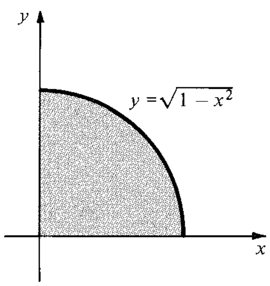


Figure 17.2.9. The region of integration for $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy\,dx$.

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dy\,dx &= \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\,dx\,dy \\
&= \int_0^1 \left[\sqrt{1-y^2}\, x \bigg|_{x=0}^{\sqrt{1-y^2}} \right] dy = \int_0^1 (1-y^2)\,dy \\
&= \left(y - \frac{y^3}{3} \right) \bigg|_0^1 = \frac{2}{3} \,.
\end{aligned}$$

Evaluating the integral in the original order requires considerably more computation! ▲

Example 5 Calculate the integral of $f(x, y) = (x+y)^2$ over the region shown in Fig. 17.2.10.

Solution In this example, there is a preferred order of integration for geometric reasons. The order $\int\int f(x, y)\,dx\,dy$, i.e., x first, requires us to break up the region into two parts by drawing the line $y = 1$; one then applies formula (5) to each part and adds the results. If we use the other order, we can cover the whole region at once:

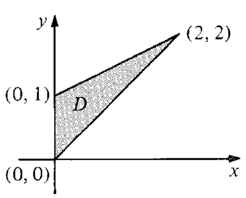


Figure 17.2.10. The region of integration for Example 5.

$$\int\int_D f(x, y)\,dx\,dy = \int_0^2 \int_x^{\frac{1}{2}x+1} f(x, y)\,dy\,dx.$$

(The lines bounding D on the bottom and top are $y = x$ and $y = \frac{1}{2}x + 1$.) The integral is thus

$$\begin{aligned}
\int_0^2 \int_x^{\frac{1}{2}x+1} (x+y)^2\,dy\,dx &= \int_0^2 \left[\frac{1}{3}(x+y)^3 \bigg|_{y=x}^{\frac{1}{2}x+1} \right] dx \\
&= \frac{1}{3} \int_0^2 \left[\left(\frac{3}{2}x + 1 \right)^3 - (2x)^3 \right] dx \\
&= \frac{1}{3} \left[\frac{1}{6} \left(\frac{3}{2}x + 1 \right)^4 \bigg|_0^2 - 2x^4 \bigg|_0^2 \right]
\end{aligned}$$

$$= \frac{1}{3}\left[\frac{1}{6}(4^4 - 1) - 2 \cdot 16\right]$$
$$= \frac{1}{3}\left[\frac{21}{2}\right] = \frac{21}{6}. \blacktriangle$$

General regions can often be broken into elementary regions, and double integrals over these regions can be computed one piece at a time.

Example 6 Find $\iint_R x^2\,dx\,dy$, where R is the shaded region in Fig. 17.2.11.

Solution Each of the regions R_1, R_2, R_3, R_4 is of types 1 and 2, so we may integrate over each one separately and sum the results. Using formula (4), we get

$$\iint_{R_1} x^2\,dx\,dy = \int_0^1\int_x^1 x^2\,dy\,dx = \int_0^1 (x^2 y)\Big|_{y=x}^{1} dx$$
$$= \int_0^1 (x^2 - x^3)\,dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1$$
$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

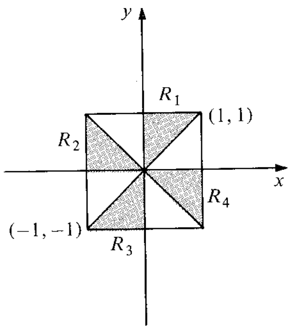


Figure 17.2.11. Integrate x^2 over the pinwheel.

Similarly, we find

$$\iint_{R_2} x^2\,dx\,dy = \int_{-1}^{0}\int_0^{-x} x^2\,dy\,dx = \frac{1}{4}.$$

By symmetry,

$$\iint_{R_3} x^2\,dx\,dy = \frac{1}{12} \quad \text{and} \quad \iint_{R_4} x^2\,dx\,dy = \frac{1}{4},$$

so

$$\iint_R x^2\,dx\,dy = \frac{1}{12} + \frac{1}{4} + \frac{1}{12} + \frac{1}{4} = \frac{2}{3}.$$

You may check that formula (5) gives the same answers. ▲

Exercises for Section 17.2

In Exercises 1–4, sketch each region and tell whether it is of type 1, type 2, both, or neither.

1. (x, y) such that $0 \leqslant y \leqslant 3x$, $0 \leqslant x \leqslant 1$.
2. (x, y) such that $y^2 \leqslant x \leqslant y$, $0 \leqslant y \leqslant 1$.
3. (x, y) such that $x^4 + y^4 \leqslant 1$.
4. (x, y) such that $\frac{1}{2} \leqslant x^4 + y^4 \leqslant 1$.

5. Find $\iint_D (x+y)^2\,dx\,dy$, where D is the shaded region in Fig. 17.2.12.

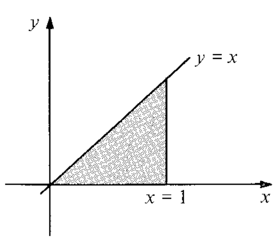


Figure 17.2.12. The region of integration for Exercises 5 and 6.

6. Find $\iint_D (1 - \sin \pi x) y\,dx\,dy$, where D is the region in Figure 17.2.12.
7. Find $\iint_D (x-y)^2 dx\,dy$, where D is the region in Figure 17.2.13.
8. Find $\iint_D y(1 - \cos(\pi x/4))\,dx\,dy$, where D is the region in Fig. 17.2.13.

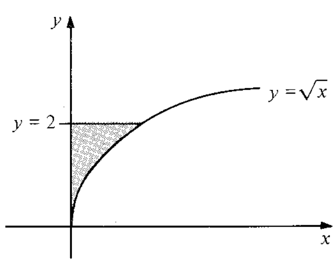


Figure 17.2.13. The region of integration for Exercises 7 and 8.

Evaluate the integrals in Exercises 9–16. Sketch and identify the type of the region (corresponding to the way the integral is written).

9. $\int_0^{\pi}\int_{\sin x}^{3\sin x} x(1+y)\,dy\,dx.$

10. $\int_0^1\int_{x-1}^{x\cos 2\pi x}(x^2+xy+1)\,dy\,dx.$

11. $\int_{-1}^1\int_{y^{2/3}}^{(2-y)^2}(y\sqrt{x}+y^3-2y)\,dx\,dy.$

12. $\int_0^2\int_{-3(\sqrt{4-x^2})/2}^{3(\sqrt{4-x^2})/2}\left(\frac{5}{\sqrt{2+x}}+y^3\right)dy\,dx.$

13. $\int_0^1\int_0^{x^2}(x^2+xy-y^2)\,dy\,dx.$

14. $\int_2^4\int_{y^2-1}^{y^3}3\,dx\,dy.$

15. $\int_0^1\int_{x^2}^{x}(x+y)^2\,dy\,dx.$

16. $\int_0^1\int_0^{3y}e^{x+y}\,dx\,dy.$

In Exercises 17–20, sketch the region of integration, interchange the order, and evaluate.

17. $\int_0^1\int_x^1 xy\,dy\,dx$

18. $\int_0^{\pi/2}\int_0^{\cos\theta}\cos\theta\,dr\,d\theta$

19. $\int_0^1\int_{1-y}^1(x+y^2)\,dx\,dy$

20. $\int_1^4\int_1^{\sqrt{x}}(x^2+y^2)\,dy\,dx$

In Exercises 21–24, integrate the given function f over the given region D.

21. $f(x,y)=x-y$; D is the triangle with vertices $(0,0)$, $(1,0)$, and $(2,1)$.

22. $f(x,y)=x^3y+\cos x$; D the triangle defined by $0\leqslant x\leqslant\pi/2$, $0\leqslant y\leqslant x$.

23. $f(x,y)=(x^2+2xy^2+2)$; D the region bounded by the graph of $y=-x^2+x$, the x axis, and the lines $x=0$ and $x=2$.

24. $f(x,y)=\sin x\cos y$; D the pinwheel in Fig. 17.2.11.

25. Show that evaluating $\iint_D dx\,dy$, where D is a region of type 1, simply reproduces the formula from Section 4.6 for the area between curves.

26. Let D be the region defined by $x^2+y^2\leqslant 1$.
  (a) Estimate $\iint_D dx\,dy$ (the area of D) within 0.1 by taking a rectangular grid in the plane and counting the number of rectangles: (i) contained entirely in D (lower sum); (ii) intersecting D (upper sum).
  (b) Compute $\iint_D dx\,dy$ exactly by using an iterated integral.

27. Which states in the United States are regions of type 1? Type 2? (Take $x=$ longitude, $y=$ latitude.)

★28. Prove: $\int_0^x\left[\int_0^t F(u)\,du\right]dt=\int_0^x(x-u)F(u)\,du.$

17.3 Applications of the Double Integral

Volumes, centers of mass, and surface areas can be calculated using double integrals.

We have observed in Section 17.1 that if $f(x,y)\geqslant 0$ on D, then the double integral $\iint_D f(x,y)\,dx\,dy$ represents the volume of the three-dimensional region R defined by (x,y) in D, $0\leqslant z\leqslant f(x,y)$. An "infinitesimal argument" for this result goes as follows. Consider R to be made of "infinitesimal rectangular prisms" with base dx and dy and height $f(x,y)$ (see Fig. 17.3.1). The total

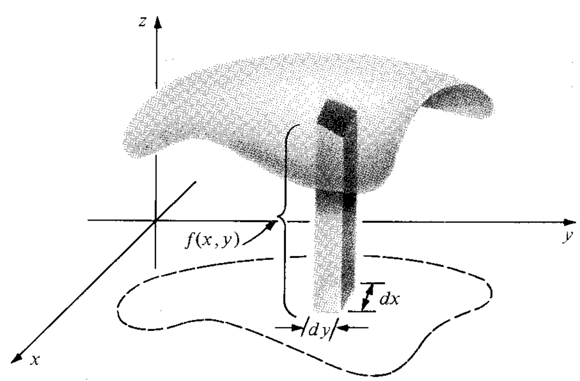


Figure 17.3.1. The region under the graph of f on D may be thought of as being composed of infinitesimal rectangular prisms.

volume is obtained by integrating (that is, "summing") the volumes of these cylinders. Notice, in particular, that if $f(x, y)$ is identically equal to 1, the volume of the region under the graph is just the area of D, so *the area of D is equal to $\iint_D dx\,dy$.*

Example 1 Compute the volume of the solid in space bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $3x + 4y = 10$, and the graph $z = x^2 + y^2$.

Solution The region is sketched in Fig. 17.3.2. Thus the volume is

$$\begin{aligned}\iint_D (x^2 + y^2)\,dx\,dy &= \int_0^{5/2}\left[\int_0^{(10-4y)/3}(x^2 + y^2)\,dx\right]dy\\ &= \int_0^{5/2}\left[\frac{(10-4y)^3}{3^4} + \frac{y^2(10-4y)}{3}\right]dy\\ &= -\frac{(10-4y)^4}{3^4\cdot 4\cdot 4} + \frac{10y^3}{9} - \frac{y^4}{3}\bigg|_0^{5/2}\\ &= \frac{10^4}{3^4\cdot 2^4} + \frac{10\cdot 5^3}{3^2\cdot 2^3} - \frac{5^4}{3\cdot 2^4} = \frac{15625}{1296} \approx 12.056. \ \blacktriangle\end{aligned}$$

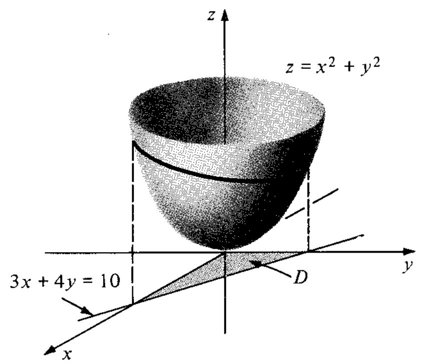


Figure 17.3.2. The volume of the region in space above D and below $z = x^2 + y^2$, is $\iint_D (x^2 + y^2)\,dx\,dy$.

(The volume in Example 1 can also be written as $5^6/(3^4 \cdot 2^4)$. Can any reader explain this simple factorization?)

By reasoning similar to that for one-variable calculus (see Section 9.3), we are led to the following definition of the average value of a function on a plane region.

Average Value

If f is an integrable function on D, the ratio of the integral to the area of D,

$$\frac{\iint_D f(x, y)\,dx\,dy}{\iint_D dx\,dy},$$

is called the *average value* of f on D.

Example 2 Find the average value of $f(x, y) = x\sin^2(xy)$ on $D = [0, \pi] \times [0, \pi]$.

Solution First we compute

$$\iint_D f(x, y)\,dx\,dy = \int_0^{\pi}\int_0^{\pi} x\sin^2(xy)\,dx\,dy$$

$$= \int_0^\pi \left(\int_0^\pi \frac{1-\cos(2xy)}{2} x\,dy \right) dx = \int_0^\pi \left(\frac{y}{2} - \frac{\sin 2(xy)}{4x} \right) x \Big|_{y=0}^{\pi} dx$$

$$= \int_0^\pi \left(\frac{\pi x}{2} - \frac{\sin(2\pi x)}{4} \right) dx = \left(\frac{\pi x^2}{4} + \frac{\cos(2\pi x)}{8\pi} \right) \Big|_0^\pi$$

$$= \frac{\pi^3}{4} + \frac{\left[\cos(2\pi^2) - 1\right]}{8\pi}.$$

Thus the average value of f is

$$\frac{\pi^3/4 + \left[\cos(2\pi^2) - 1\right]/8\pi}{\pi^2} = \frac{\pi}{4} + \frac{\cos(2\pi^2) - 1}{8\pi^3} \approx 0.7839. \blacktriangle$$

Double integration also allows us to find the center of mass of a plate with variable density. Let D represent a plate with variable density $\rho(x, y)$. We can imagine breaking D into infinitesimal elements with mass $\rho(x, y)\,dx\,dy$; the total mass is thus $\int\int_D \rho(x, y)\,dx\,dy$. Applying the consolidation principle (see Section 9.4) to the infinitesimal rectangles, one can derive the formulas in the following box.

Center of Mass

$$\bar{x} = \frac{\int\int_D x\rho(x, y)\,dx\,dy}{\int\int_D \rho(x, y)\,dx\,dy} \quad \text{and} \quad \bar{y} = \frac{\int\int_D y\rho(x, y)\,dx\,dy}{\int\int_D \rho(x, y)\,dx\,dy}.$$

Example 3 Find the center of mass of the rectangle $[0, 1] \times [0, 1]$ if the density is e^{x+y}.

Solution First we compute the mass:

$$\int\int_D e^{x+y}\,dx\,dy = \int_0^1 \int_0^1 e^{x+y}\,dx\,dy = \int_0^1 \left(e^{x+y}\big|_{x=0}^1\right) dy$$

$$= \int_0^1 (e^{1+y} - e^y)\,dy$$

$$= e^{1+y} - e^y\big|_{y=0}^1 = e^2 - e - (e-1) = e^2 - 2e + 1.$$

The numerator in the formula for $\bar{x}$ is

$$\int_0^1 \int_0^1 xe^{x+y}\,dx\,dy = \int_0^1 (xe^{x+y} - e^{x+y})\Big|_{x=0}^1 dy$$

$$= \int_0^1 \left[e^{1+y} - e^{1+y} - (0e^y - e^y) \right] dy$$

$$= \int_0^1 e^y\,dy = e^y\big|_{y=0}^1 = e - 1,$$

so

$$\bar{x} = \frac{e-1}{e^2 - 2e + 1} = \frac{e-1}{(e-1)^2} = \frac{1}{e-1} \approx 0.582.$$

The roles of x and y may be interchanged in all these calculations, so $\bar{y} = 1/(e-1) \approx 0.582$ as well. $\blacktriangle$

In Section 10.3 we used ordinary integration to determine the area of surfaces of revolution. Using the double integral, we can find the area of general curved surfaces. We confine ourselves here to an infinitesimal argument—the rigorous theory of surface area is quite subtle.[2]

To find the area of the graph $z = f(x, y)$ of a function f over the plane region D, we divide D into "infinitesimal rectangles" which are of the form $[x, x + dx] \times [y, y + dy]$. The image of this infinitesimal rectangle on the graph of f is approximately an "infinitesimal parallelogram" with vertices at

$$
\begin{aligned}
P_1 &= (x, y, f(x, y)),\\
P_2 &= (x + dx, y, f(x + dx, y)) \approx (x + dx, y, f(x, y) + f_x(x, y)\,dx),\\
P_3 &= (x, y + dy, f(x, y + dy)) \approx (x, y + dy, f(x, y) + f_y(x, y)\,dy),\\
P_4 &= (x + dx, y + dy, f(x + dx, y + dy))\\
&\approx (x + dx, y + dy, f(x, y) + f_x(x, y)\,dx + f_y(x, y)\,dy).
\end{aligned}
$$

(See Fig. 17.3.3.)

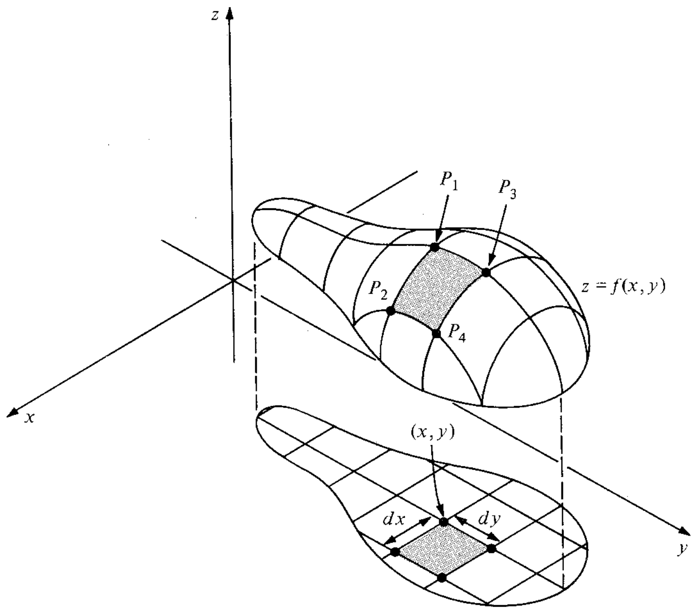


Figure 17.3.3. The "image" on the surface $z = f(x, y)$ of an infinitesimal rectangle in the plane is the infinitesimal parallelogram $P_1P_2P_4P_3$.

We compute the area dA of this parallelogram by taking the length of the cross product of the vectors from P_1 to P_2 and from P_1 to P_3 (see Section 13.5). The vectors in question are $dx\,\mathbf{i} + f_x(x, y)\,dx\,\mathbf{k}$ and $dy\,\mathbf{j} + f_y(x, y)\,dy\,\mathbf{k}$; their cross product is

$$
\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dx & 0 & f_x(x, y)\,dx \\ 0 & dy & f_y(x, y)\,dy \end{vmatrix} = -f_x(x, y)\,dx\,dy\,\mathbf{i} - f_y(x, y)\,dx\,dy\,\mathbf{j} + dx\,dy\,\mathbf{k},
$$

and the length of this vector is $dA = \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\,dx\,dy$. To get the area of the surface, we "sum" the areas of the infinitesimal parallelograms by integrating over D.

[2] See T. Radó, *Length and Area*, American Mathematical Society Colloquium Publications, Volume 30 (1958).

Surface Area of a Graph

$$\text{Area} = \iint_D dA = \iint_D \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,dx\,dy$$
$$= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy.$$

Note the similarity of this expression with the formula for the arc length of a graph (Section 10.3). As with arc length, the square root makes the analytic evaluation of surface area integrals difficult or even impossible in all but a few accidentally simple cases.

Example 4 Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 1$ lying above the ellipse $x^2 + (y^2/a^2) \leqslant 1$; ($a$ is a constant satisfying $0 < a \leqslant 1$).

Solution The region described by $x^2 + (y^2/a^2) \leqslant 1$ is type 1 with $\phi_1(x) = -a\sqrt{1-x^2}$ and $\phi_2(x) = a\sqrt{1-x^2}$; $-1 \leqslant x \leqslant 1$. The upper hemisphere may be described by the equation $z = f(x,y) = \sqrt{1-x^2-y^2}$. The partial derivatives of f are $\partial z/\partial x = -x/\sqrt{1-x^2-y^2}$ and $\partial z/\partial y = -y/\sqrt{1-x^2-y^2}$, so the area integrand is

$$\sqrt{1 + \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2}} = \frac{1}{\sqrt{1-x^2-y^2}}$$

and the area is

$$A = \iint_D \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \int_{-1}^{1}\left[\int_{-a\sqrt{1-x^2}}^{a\sqrt{1-x^2}} \frac{dy}{\sqrt{1-x^2-y^2}}\right]dx$$
$$= \int_{-1}^{1}\left(\sin^{-1}\frac{y}{\sqrt{1-x^2}}\,\Bigg|_{-a\sqrt{1-x^2}}^{a\sqrt{1-x^2}}\right)dx = 2\int_{-1}^{1}\sin^{-1}a\,dx = 4\sin^{-1}a.$$

(See Fig. 17.3.4.) As a check on our answer, note that if $a = 1$, we get $4\sin^{-1}1 = 4\cdot\pi/2 = 2\pi$, the correct formula for the area of a hemisphere of radius 1 (the surface area of a full sphere of radius r is $4\pi r^2$). ▲

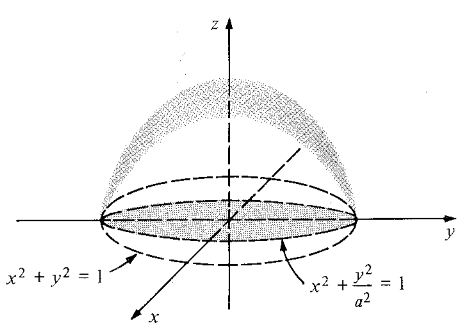


Figure 17.3.4. The area of the hemisphere above the ellipse $x^2 + y^2/a^2 \leqslant 1$ is $4\sin^{-1}a$.

Example 5 The equation in xyz space of the surface obtained by revolving the graph $y = f(x)$ about the x axis is $y^2 + z^2 = [f(x)]^2$. Express as a double integral the area of the part of this surface lying between the planes $x = a$ and $x = b$. Carry out the integration over y. Do you recognize the resulting integral over x?

Solution Writing z as a function of x and y, we have $z = g(x, y) = \pm\sqrt{f(x)^2 - y^2}$. The domain of this function consists of those (x, y) with $-f(x) \leqslant y \leqslant f(x)$, so the surface in question lies over the type 1 region D defined by $a \leqslant x \leqslant b$, $-f(x) \leqslant y \leqslant f(x)$. (See Fig. 17.3.5.) The partial derivatives of g are

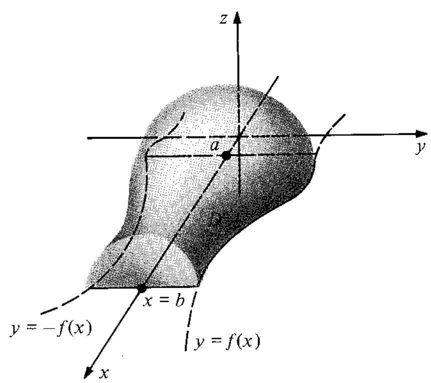


Figure 17.3.5. Finding the surface area when $y = f(x)$ is rotated about the x axis.

$$g_x(x, y) = f'(x)f(x)/\sqrt{f(x)^2 - y^2}\,, \qquad g_y(x, y) = -y/\sqrt{f(x)^2 - y^2}\,,$$

so the surface area integrand is

$$\sqrt{1 + \frac{f'(x)^2 f(x)^2}{f(x)^2 - y^2} + \frac{y^2}{f(x)^2 - y^2}} = \sqrt{\frac{f(x)^2 - y^2 + f'(x)^2 f(x)^2 + y^2}{f(x)^2 - y^2}}$$

$$= f(x)\sqrt{\frac{1 + f'(x)^2}{f(x)^2 - y^2}}$$

and the area is $A = 2\int_a^b \int_{-f(x)}^{f(x)} f(x)\sqrt{\dfrac{1 + f'(x)^2}{f(x)^2 - y^2}}\, dy\, dx.$

(The factor of 2 occurs since half the surface lies below the xy plane.) We can carry out the integration over y:

$$A = 2\int_a^b f(x)\sqrt{1 + f'(x)^2}\left[\int_{-f(x)}^{f(x)} \frac{1}{\sqrt{f(x)^2 - y^2}}\, dy\right] dx$$

$$= 2\int_a^b f(x)\sqrt{1 + f'(x)^2}\left[\sin^{-1}\left(\frac{y}{f(x)}\right)\Bigg|_{y=-f(x)}^{f(x)}\right] dx$$

$$= 2\int_a^b f(x)\sqrt{1 + f'(x)^2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) dx$$

$$= 2\pi\int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx.$$

This is the formula for the area of a surface of revolution (Section 10.3). ▲

Exercises for Section 17.3

In Exercises 1–4, find the volume under the graph of $f(x, y)$ between the given planes $x = a$, $x = b$, $y = c$, and $y = d$,

1. $f(x, y) = x \sin y + 3$; $a = 0$, $b = 2$, $c = \pi$, $d = 3\pi$;
2. $f(x, y) = xy^5 + 2x^4 + 6$; $a = 0$, $b = 1$, $c = -1$, $d = 1$.
3. $f(x, y) = 2x^4 + 3y^{2/3}$; $a = -1$, $b = 1$, $c = 0$, $d = 2$;
4. $f(x, y) = xy\sqrt{x^2 + 3y^2}$; $a = 1$, $b = 2$, $c = 1$, $d = 2$.
5. Compute the volume under the graph of $f(x, y) = 1 + \sin(\pi y/2) + x$ on the parallelogram in the xy plane with vertices $(0,0), (1,2), (2,0), (3,2)$. Sketch.
6. Compute the volume under the graph of $f(x, y) = 4x^2 + 3y^2 + 27$ on the disk of radius 2 centered at $(0, 1)$. Sketch.
7. Compute the volume under the graph of $f(x, y) = (\cos y)e^{1-\cos 2x} + xy$ on the region bounded by the line $y = 2x$, the x axis, and the line $x = \pi/2$.
8. Compute the volume between the graphs of the functions $f(x, y) = 2x + 1$ and $g(x, y) = -x - 3y - 6$ on the region bounded by the y axis and the curve $x = 4 - y^2$. Sketch.
9. Find the volume of the region bounded by the planes $x = 0$ and $z = 0$ and the surfaces $x = -4y^2 + 3$, and $z = x^3y$.
10. Find the volume of the region bounded by the planes $x = 1$, $z = 0$, $y = x + 1$, $y = -x - 1$ and the surface $z = 2x^2 + y^4$.

In Exercises 11–14, find the average value of the given function on the given region.

11. $f(x, y) = y \sin xy$; $D = [0, \pi] \times [0, \pi]$.
12. $f(x, y) = x^2 + y^2$; $D =$ the ring between the circles $x^2 + y^2 = \frac{1}{2}$ and $x^2 + y^2 = 1$.
13. $f(x, y) = e^{x+y}$; $D =$ the triangle with vertices at $(0,0)$, $(0, 1)$, and $(1, 0)$.
14. $f(x, y) = 1/(x + y)$; $D = [e, e^2] \times [e, e^2]$.

Find the average value of $x^2 + y^2$ over each of the regions in Exercises 15–18.

15. The square $[0, 1] \times [0, 1]$.
16. The square $[a, a + 1] \times [0, 1]$, where $a > 0$.
17. The square $[0, a] \times [0, a]$, where $a > 0$.
18. The set of (x, y) such that $x^2 + y^2 < a^2$.
19. Find the center of mass of the region between $y = x^2$ and $y = x$ if the density is $x + y$.
20. Find the center of mass of the region between $y = 0$, $y = x^2$, where $0 \leqslant x \leqslant \frac{1}{2}$.
21. Find the center of mass of the disk determined by $(x - 1)^2 + y^2 \leqslant 1$ if the density is x^2.
22. Repeat Exercise 21 if the density is y^2.
23. Find the area of the graph of the function $f(x, y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ which lies over the domain $D = [0, 1] \times [0, 1]$.
24. Find the area cut out of the cylinder $x^2 + z^2 = 1$ by the cylinder $x^2 + y^2 = 1$.
25. Calculate the area of the part of the cone $z^2 = x^2 + y^2$ lying in the region of space defined by $x \geqslant 0, y \geqslant 0, z \leqslant 1$.
26. Find the area of the portion of the cylinder $x^2 + z^2 = 4$ which lies above the rectangle defined by $-1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 2$.
27. Show that if a plate D has constant density, then the average values of x and y on D are the coordinates of the center of mass.
28. Find the center of mass of the region (composed of two pieces) bounded by $y = x^3$ and $y = \sqrt[3]{x}$ if the density is $(x - y)^2$. Try to minimize your work by exploiting some symmetry in the problem.
29. (a) Prove that the area on a sphere of radius r cut out by a cone of angle ϕ is $2\pi r^2(1 - \cos\phi)$ (Fig. 17.3.6).

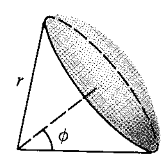


Figure 17.3.6. The area of the cap is $2\pi r^2(1 - \cos\phi)$.

(b) A sphere of radius 1 sits with its center on the surface of a sphere of radius $r > 1$. Show that the area of surface on the second sphere cut out by the first sphere is π. (Does something about this result surprise you?)

30. A uniform rectangular steel plate of sides a and b rotates about its center of gravity with constant angular velocity ω.
    (a) Kinetic energy equals $\frac{1}{2}$(mass)(velocity)^2. Argue that the kinetic energy of any element of mass $\rho\, dx\, dy$ ($\rho =$ constant) is given by $\rho(\omega^2/2)(x^2 + y^2)\, dx\, dy$, provided the origin $(0, 0)$ is placed at the center of gravity of the plate.
    (b) Justify the formula for kinetic energy:
$$\text{K.E.} = \iint_{\text{plate}} \rho \frac{\omega^2}{2}(x^2 + y^2)\, dx\, dy.$$
    (c) Evaluate the integral, assuming that the plate is described by $-a/2 \leqslant x \leqslant a/2$, $-b/2 \leqslant y \leqslant b/2$.

31. A sculptured gold plate D is defined by $0 \leqslant x \leqslant 2\pi$ and $0 \leqslant y \leqslant \pi$ (centimeters) and has mass density $\rho(x, y) = y^2\sin^2 4x + 2$ (grams per square centimeter). If gold sells for \$7 per gram, how much is the gold in the plate worth?

★32. (a) Relate the integrand in the surface area formula to the angle between **k** and the normal to the surface $z = f(x, y)$.
(b) Express the ratio of the area of the graph of f over D to the area of D as the average value of some geometrically defined quantity.

★33. Express as a double integral the volume enclosed by the surface of revolution in Example 5. Carry out the integration over y and show that the resulting integral over x is a formula in Section 9.1.

★34. Let n right circular cylinders of radius r intersect such that their axes lie in a plane, meeting at one point with equal angles. Find the volume of their intersection.

17.4 Triple Integrals

Integrals over regions in three-dimensional space require the triple integral.

The basic ideas developed in Sections 17.1 and 17.2 can be readily extended from double to triple integrals. As with double integrals, one of the most powerful evaluation methods is reduction to iterated integrals. A second important technique, which we discuss in Section 17.5, is the method of changing variables.

If the temperature inside an oven is not uniform, determining the average temperature involves "summing" the values of the temperature function at all points in the solid region enclosed by the oven walls. Such a sum is expressed mathematically as a triple integral.

We formalize the ideas just as we did for double integrals. Suppose that W is a box (that is, rectangular parallelepiped) in space bounded by the planes $x = a$, $x = b$, $y = c$, $y = d$, and $z = p$, $z = q$, as in Fig. 17.4.1. We denote this

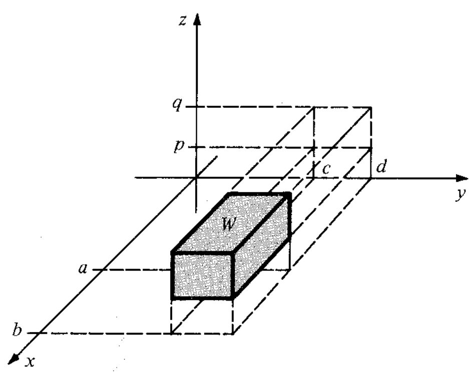


Figure 17.4.1. The box $W = [a, b] \times [c, d] \times [p, q]$ consists of points (x, y, z) satisfying $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, and $p \leqslant z \leqslant q$.

box by $[a, b] \times [c, d] \times [p, q]$. Let $f(x, y, z)$ be a function defined for (x, y, z) in W—that is, for

$$a \leqslant x \leqslant b, \qquad c \leqslant y \leqslant d, \qquad p \leqslant z \leqslant q.$$

In order to define the *triple integral*

$$\iiint_W f(x, y, z)\,dx\,dy\,dz,$$

we first define the concept of a step function of three variables.

A function $g(x, y, z)$ defined on $[a,b]\times[c,d]\times[p,q]$ is called a *step function* if there are partitions

$$a = t_0 < t_1 < \cdots < t_n = b \quad \text{of} \quad [a,b],$$
$$c = s_0 < s_1 < \cdots < s_m = d \quad \text{of} \quad [c,d],$$
$$p = r_0 < r_1 < \cdots < r_l = q \quad \text{of} \quad [p,q]$$

such that $g(x, y, z)$ has the constant value k_{ijk} for (x, y, z) in the open box

$$W_{ijk} = (t_{i-1}, t_i) \times (s_{j-1}, s_j) \times (r_{k-1}, r_k).$$

We cannot draw the graphs of functions of three variables; however, we can indicate the value k_{ijk} associated with each box (see Fig. 17.4.2). The

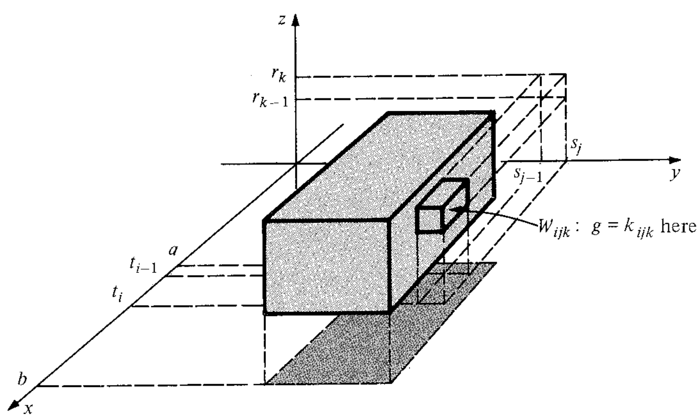


Figure 17.4.2. g has the value k_{ijk} on the small box W_{ijk}.

integral of g is defined as a sum of *nml* terms:

$$\iiint_W g(x, y, z)\,dx\,dy\,dz = \sum_{i=1,j=1,k=1}^{n,m,l} k_{ijk}(\text{volume } W_{ijk})$$
$$= \sum_{i=1,j=1,k=1}^{n,m,l} k_{ijk}(\Delta t_i)(\Delta s_j)(\Delta r_k).$$

If f is any function on W, its lower (respectively upper) sums are defined, as before, as the integrals of step functions g (respectively h) such that $g(x, y, z) \leqslant f(x, y, z)$ (respectively $h(x, y, z) \geqslant f(x, y, z)$) for all points (x, y, z) in W.

The Triple Integral

We say that f is *integrable* on W if there is a number S_0 such that every $S < S_0$ is a lower sum for f on W and every $S > S_0$ is an upper sum. The number S_0 is called the *integral* of f over W and is denoted by

$$\iiint_W f(x, y, z)\,dx\,dy\,dz.$$

At this point you should look back at the basic properties of the double integrals listed in Section 17.1. Similar properties hold for triple integrals. Furthermore, there is a similar reduction to iterated integrals.

Theorem: Reduction to Iterated Integrals

Let $f(x, y, z)$ be integrable on the box $W = [a, b] \times [c, d] \times [p, q]$. Then any iterated integral which exists is equal to the triple integral; that is,

$$\iiint_W f(x, y, z)\, dx\, dy\, dz = \int_p^q \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

$$= \int_p^q \int_a^b \int_c^d f(x, y, z)\, dy\, dx\, dz$$

$$= \int_a^b \int_p^q \int_c^d f(x, y, z)\, dy\, dz\, dx,$$

and so on. (There are six possible orders altogether.)

The proof of this result is just like the corresponding one in Section 17.1, so we omit it.

Example 1 (a) Let W be the box $[0, 1] \times [-\frac{1}{2}, 0] \times [0, \frac{1}{3}]$. Evaluate

$$\iiint_W (x + 2y + 3z)^2\, dx\, dy\, dz.$$

(b) Verify that we get the same answer if the integration is done in the order y first, then z, and then x.

Solution (a) According to the reduction to iterated integrals, this integral may be evaluated as

$$\int_0^{1/3} \int_{-1/2}^0 \int_0^1 (x + 2y + 3z)^2\, dx\, dy\, dz$$

$$= \int_0^{1/3} \int_{-1/2}^0 \left[\left. \frac{(x + 2y + 3z)^3}{3} \right|_{x=0}^1 \right] dy\, dz$$

$$= \int_0^{1/3} \int_{-1/2}^0 \frac{1}{3} \left[(1 + 2y + 3z)^3 - (2y + 3z)^3 \right] dy\, dz$$

$$= \int_0^{1/3} \frac{1}{24} \left[(1 + 2y + 3z)^4 - (2y + 3z)^4 \right] \Big|_{y=-1/2}^0 dz$$

$$= \int_0^{1/3} \frac{1}{24} \left[(3z + 1)^4 - 2(3z)^4 + (3z - 1)^4 \right] dz$$

$$= \frac{1}{24 \cdot 15} \left[(3z + 1)^5 - 2(3z)^5 + (3z - 1)^5 \right] \Big|_{z=0}^{1/3}$$

$$= \frac{1}{24 \cdot 15} (2^5 - 2) = \frac{1}{12}.$$

(b) $$\iiint_W (x + 2y + 3z)^2\, dy\, dz\, dx$$

$$= \int_0^1 \int_0^{1/3} \int_{-1/2}^0 (x + 2y + 3z)^2\, dy\, dz\, dx$$

$$= \int_0^1 \int_0^{1/3} \left[\frac{(x+2y+3z)^3}{6} \Bigg|_{y=-1/2}^{0} \right] dz\,dx$$

$$= \int_0^1 \int_0^{1/3} \frac{1}{6} \left[(x+3z)^3 - (x+3z-1)^3 \right] dz\,dx$$

$$= \int_0^1 \frac{1}{6} \left[\left(\frac{(x+3z)^4}{12} - \frac{(x+3z-1)^4}{12} \right) \Bigg|_{z=0}^{1/3} \right] dx$$

$$= \int_0^1 \frac{1}{72} \left[(x+1)^4 + (x-1)^4 - 2x^4 \right] dx$$

$$= \frac{1}{72}\,\frac{1}{5} \left[(x+1)^5 + (x-1)^5 - 2x^5 \right]_{x=0}^{1} = \frac{1}{12}. \blacktriangle$$

Example 2 Evaluate the integral of e^{x+y+z} over the box $[0,1] \times [0,1] \times [0,1]$.

Solution

$$\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\,dx\,dy\,dz = \int_0^1 \int_0^1 \left(e^{x+y+z}\big|_{x=0}^{1} \right) dy\,dz$$

$$= \int_0^1 \int_0^1 \left(e^{1+y+z} - e^{y+z} \right) dy\,dz = \int_0^1 \left[e^{1+y+z} - e^{y+z} \right]_{y=0}^{1} dz$$

$$= \int_0^1 \left[e^{2+z} - 2e^{1+z} + e^z \right] dz = \left[e^{2+z} - 2e^{1+z} + e^z \right]_0^1$$

$$= e^3 - 3e^2 + 3e - 1 = (e-1)^3. \blacktriangle$$

As in the two-variable case, we define the integral of a function f over a bounded region W by defining a new function f^*, equal to f on W and zero outside W, and then setting

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f^*(x,y,z)\,dx\,dy\,dz,$$

where W^* is any box containing the region W.

As before, we restrict our attention to particularly simple regions. A three-dimensional region W will be said to be of *type* I if there is an elementary region D in the xy plane and a pair of continuous functions, $\gamma_1(x,y)$ and $\gamma_2(x,y)$ defined on D, such that W consists of those triples (x,y,z) for which $(x,y) \in D$ and $\gamma_1(x,y) \leqslant z \leqslant \gamma_2(x,y)$. The region D may itself be of type 1 or type 2, so there are two possible descriptions of a type I region:

$$\begin{aligned} &a \leqslant x \leqslant b, \qquad \phi_1(x) \leqslant y \leqslant \phi_2(x), \\ &\gamma_1(x,y) \leqslant z \leqslant \gamma_2(x,y) \qquad \text{(if } D \text{ is of type 1)} \end{aligned} \tag{1}$$

or

$$\begin{aligned} &c \leqslant y \leqslant d, \qquad \psi_1(y) \leqslant x \leqslant \psi_2(y), \\ &\gamma_1(x,y) \leqslant z \leqslant \gamma_2(x,y) \qquad \text{(if } D \text{ is of type 2).} \end{aligned} \tag{2}$$

Figure 17.4.3 on the next page shows two regions of type I that are described by conditions (1) and (2), respectively.

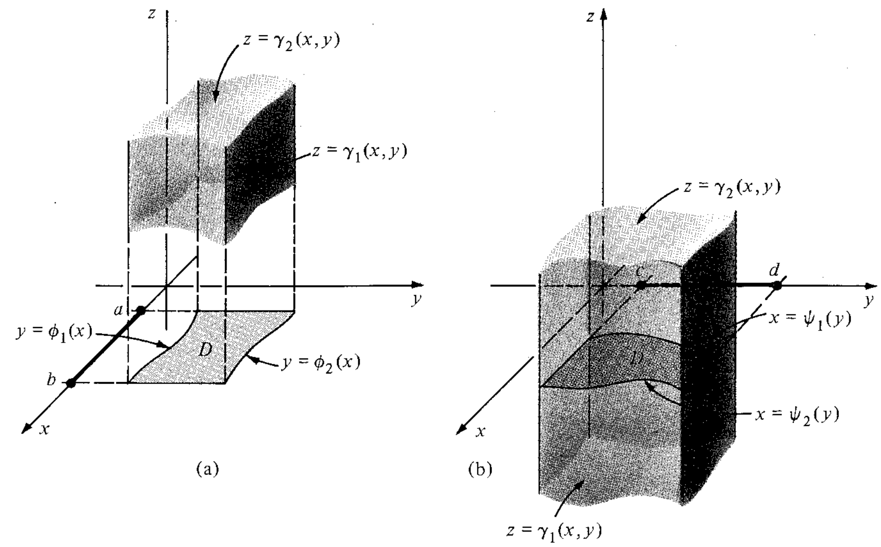


Figure 17.4.3. A region of type I lies between two graphs $z = \gamma_1(x, y)$ and $z = \gamma_2(x, y)$.

A region W is of *type II* if it can be expressed in form (1) or (2) with the roles of x and z interchanged, and W is of *type III* if it can be expressed in form (1) or (2) with y and z interchanged. See Fig. 17.4.4.

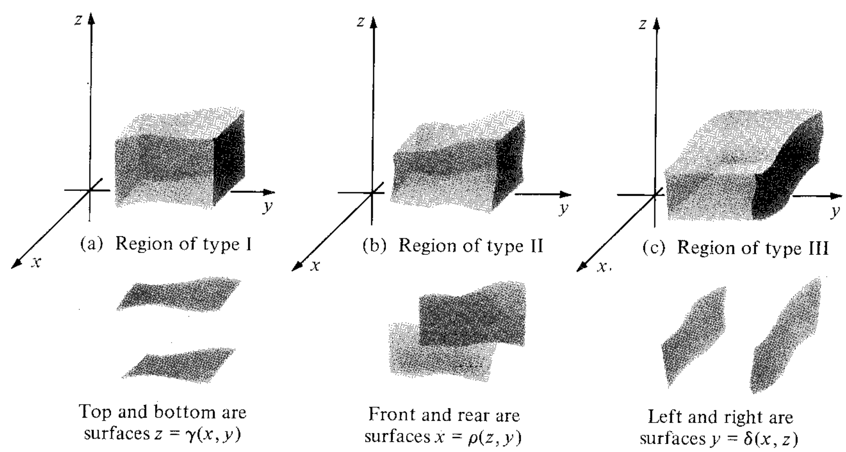


Figure 17.4.4. The three types of regions in space.

Notice that a given region may be of two or even three types at once. (See Fig. 17.4.5.) As with regions in the plane, we call a region of type I, II, or III in space an *elementary region*.

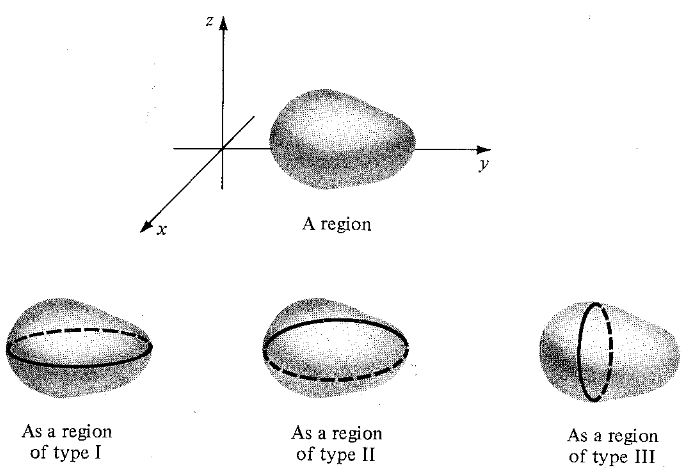


Figure 17.4.5. Regions in space can be of more than one type. This one is of all three types.

Example 3 Show that the unit ball $x^2 + y^2 + z^2 \leqslant 1$ is a region of all three types.

Solution As a type I region, we can express it as

$$-1 \leqslant x \leqslant 1,$$
$$-\sqrt{1-x^2} \leqslant y \leqslant \sqrt{1-x^2}\,,$$
$$-\sqrt{1-x^2-y^2} \leqslant z \leqslant \sqrt{1-x^2-y^2}\,.$$

In doing this, we first write the top and bottom hemispheres as $z = \sqrt{1-x^2-y^2}$ and $z = -\sqrt{1-x^2-y^2}$, where x and y vary over the unit disk (that is, $-\sqrt{1-x^2} \leqslant y \leqslant \sqrt{1-x^2}$ and x varies between -1 and 1). (See Fig. 17.4.6.) We write the region as a type II or III region in a similar manner by interchanging the roles of x, y, and z in the defining inequalities. ▲

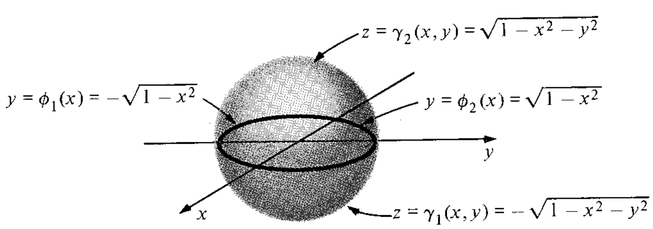


Figure 17.4.6. The unit ball described as a region of type I.

As with integrals in the plane, any function of three variables which is continuous over an elementary region is integrable on that region. An argument like that for double integrals shows that a triple integral over an elementary region can be rewritten as an iterated integral in which the limits of integration are functions. The formulas for such iterated integrals are given in the following display.

Triple Integrals

Suppose that W is of type I. Then either

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dz\,dy\,dx \qquad (3)$$

(see Fig. 17.4.3(a)) or

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dz\,dx\,dy \qquad (4)$$

(see Fig. 17.4.3(b)).

If W is of type II, it can be expressed as the set of all (x, y, z) such that

$$a \leqslant z \leqslant b, \qquad \phi_1(z) \leqslant y \leqslant \phi_2(z), \qquad \rho_1(z,y) \leqslant x \leqslant \rho_2(z,y).$$

Then

$$\iiint_W f(x, y, z)\,dx\,dy\,dz = \int_a^b \int_{\phi_1(z)}^{\phi_2(z)} \int_{\rho_1(z,y)}^{\rho_2(z,y)} f(x, y, z)\,dx\,dy\,dz. \tag{5}$$

If W is expressed as the set of all (x, y, z) such that

$$c \leqslant y \leqslant d, \qquad \psi_1(y) \leqslant z \leqslant \psi_2(y), \qquad \rho_1(z, y) \leqslant x \leqslant \rho_2(z, y),$$

then

$$\iiint_W f(x, y, z)\,dx\,dy\,dz = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \int_{\rho_1(z,y)}^{\rho_2(z,y)} f(x, y, z)\,dx\,dz\,dy. \tag{6}$$

There are similar formulas for type III regions (Exercise 22).

Another way to write formula (3) is

$$\iiint_W f(x, y, z)\,dx\,dy\,dz = \iint_D \left[\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)\,dz \right] dy\,dx,$$

and for formula (4),

$$\iiint_W f(x, y, z)\,dx\,dy\,dz = \iint_D \left[\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)\,dz \right] dx\,dy.$$

Notice that the triple integral $\iiint_W dx\,dy\,dz$ is simply the volume of W.

Example 4 Verify the formula for the volume of a ball of radius 1: $\iiint_W dx\,dy\,dz = \frac{4}{3}\pi$, where W is the set of (x, y, z) with $x^2 + y^2 + z^2 \leqslant 1$.

Solution As explained in Example 3, the ball is a region of type I. By formula (3), the integral is

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx.$$

Holding y and x fixed and integrating with respect to z yields

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[z \Big|_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \right] dy\,dx$$

$$= 2\int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)^{1/2}\,dy \right] dx.$$

Since x is fixed in the integral over y, this integral can be expressed as $\int_{-a}^{a} (a^2 - y^2)^{1/2}\,dy$, where $a = (1 - x^2)^{1/2}$. This integral represents the area of a semicircular region of radius a, so that

$$\int_{-a}^{a} (a^2 - y^2)^{1/2}\,dy = \frac{a^2}{2}\pi.$$

(We could also have used a trigonometric substitution.) Thus

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)^{1/2}\,dy = \frac{1 - x^2}{2}\pi,$$

and so

$$2\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)^{1/2}\,dy\,dx = 2\int_{-1}^{1} \pi \frac{1 - x^2}{2}\,dx$$

$$= \int_{-1}^{1} \pi(1 - x^2)\,dx = \pi\left(x - \frac{x^3}{3}\right)\Big|_{x=-1}^{1} = \frac{4}{3}\pi. \ \blacktriangle$$

Example 5 Let W be the region bounded by the planes $x = 0$, $y = 0$, and $z = 2$, and the surface $z = x^2 + y^2$. Compute $\iiint_W x\,dx\,dy\,dz$ and sketch the region.

Solution *Method* 1. The region W is sketched in Fig. 17.4.7. We may write this as a region of type I with $\gamma_1(x, y) = x^2 + y^2$, $\gamma_2(x, y) = 2$, $\phi_1(x) = 0$, $\phi_2(x) = \sqrt{2 - x^2}$, $a = 0$, and $b = \sqrt{2}$. By formula (3),

$$\iiint_W x\,dx\,dy\,dz = \int_0^{\sqrt{2}}\left[\int_0^{\sqrt{2-x^2}}\left(\int_{x^2+y^2}^{2} x\,dz\right)dy\right]dx$$

$$= \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} x(2 - x^2 - y^2)\,dy\,dx$$

$$= \int_0^{\sqrt{2}} x\left[(2 - x^2)^{3/2} - \frac{(2-x^2)^{3/2}}{3}\right]dx$$

$$= \int_0^{\sqrt{2}} \frac{2x}{3}(2 - x^2)^{3/2}\,dx = \left.\frac{-2(2-x^2)^{5/2}}{15}\right|_0^{\sqrt{2}}$$

$$= 2 \cdot \frac{2^{5/2}}{15} = \frac{8\sqrt{2}}{15}.$$

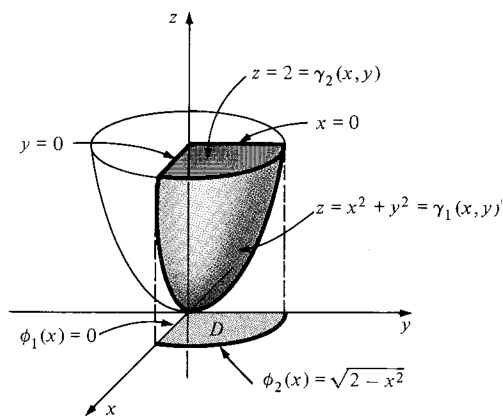


Figure 17.4.7. W is the region below the plane $z = 2$, above the paraboloid $z = x^2 + y^2$, and on the positive sides of the planes $x = 0$, $y = 0$.

Method 2. W can be expressed as the set of (x, y, z) with the property that $\rho_1(z, y) = 0 \leqslant x \leqslant (z - y^2)^{1/2} = \rho_2(z, y)$ and (z, y) in D, where D is the subset of the yz plane with $0 \leqslant z \leqslant 2$ and $0 \leqslant y \leqslant z^{1/2}$ (see Fig. 17.4.8). Therefore,

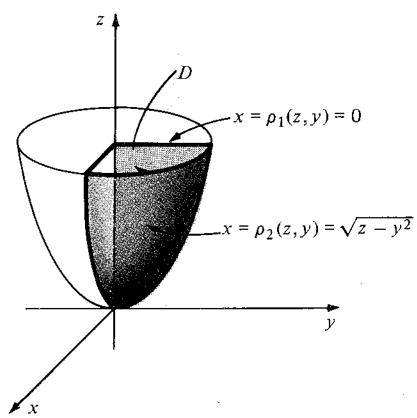


Figure 17.4.8. W as a type II region.

$$\iiint_W x\,dx\,dy\,dz = \iint_D \left(\int_{\rho_1(z,y)}^{\rho_2(z,y)} x\,dx \right) dy\,dz$$

$$= \int_0^2 \left[\int_0^{z^{1/2}} \left(\int_0^{(z-y^2)^{1/2}} x\,dx \right) dy \right] dz = \int_0^2 \int_0^{z^{1/2}} \left(\frac{z-y^2}{2} \right) dy\,dz$$

$$= \frac{1}{2} \int_0^2 \left(z^{3/2} - \frac{z^{3/2}}{3} \right) dz = \frac{1}{2} \int_0^2 \frac{2}{3} z^{3/2}\,dz$$

$$= \left[\frac{2}{15} z^{5/2} \right]_0^2 = \frac{2}{15} 2^{5/2} = \frac{8\sqrt{2}}{15},$$

which agrees with our other answer. ▲

Example 6 Evaluate $\int_0^1 \int_0^x \int_{x^2+y^2}^1 dz\,dy\,dx$. Sketch the region W of integration and interpret.

Solution

$$\int_0^1 \int_0^x \int_{x^2+y^2}^1 dz\,dy\,dx = \int_0^1 \int_0^x (1 - x^2 - y^2)\,dy\,dx$$

$$= \int_0^1 \left(x - x^3 - \frac{x^3}{3} \right) dx = \frac{1}{2} - \frac{1}{4} - \frac{1}{12} = \frac{1}{6}.$$

This is the volume of the region sketched in Fig. 17.4.9. ▲

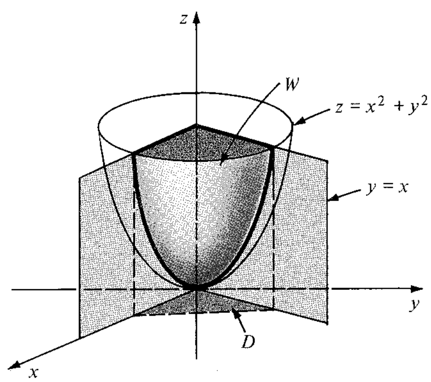


Figure 17.4.9. The region W (type I) for Example 6.

Exercises for Section 17.4

1. Evaluate $\iiint_W (2x + 3y + z)\,dx\,dy\,dz$, where $W = [1,2] \times [-1,1] \times [0,1]$, in at least two ways.
2. Evaluate the integral $\iiint_W x^2\,dx\,dy\,dz$, where $W = [0,1] \times [-1,1] \times [0,1]$, in at least two ways.
3. Integrate the function $\sin(x + y + z)$ over the box $[0,\pi] \times [0,\pi] \times [0,\pi]$.
4. Integrate ze^{x+y} over $[0,1] \times [0,1] \times [0,1]$.

Determine whether each of the regions in Exercises 5–8 is of type I, II, or III.

5. The region between the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$.
6. The region cut out of the ball $x^2 + y^2 + z^2 \leqslant 4$ by the elliptic cylinder $2x^2 + z^2 = 1$ i.e. the region inside the cylinder and the ball.
7. The region inside the ellipsoid $x^2 + 2y^2 + z^2 = 1$ and above the plane $z = 0$.
8. The region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 4$, and $x = z - y - 1$.

Find the volumes of the regions in Exercises 9–12.

9. The region bounded by $z = x^2 + y^2$ and $z = 10 - x^2 - 2y^2$.
10. The solid bounded by $x^2 + 2y^2 = 2$, $z = 0$, and $x + y + 2z = 2$.
11. The solid bounded by $x = y$, $z = 0$, $y = 0$, $x = 1$, and $x + y + z = 0$.
12. The region common to the intersecting cylinders $x^2 + y^2 \leqslant a^2$ and $x^2 + z^2 \leqslant a^2$.

Evaluate the integrals in Exercises 13–20.

13. $\int_0^1\int_1^2\int_2^3 \cos[\pi(x+y+z)]\,dx\,dy\,dz$.
14. $\int_0^1\int_0^x\int_0^y (y+xz)\,dz\,dy\,dx$.
15. $\int\int\int_R (x^2+y^2+z^2)\,dx\,dy\,dz$; R is the region bounded by $x+y+z=a$ (where $a>0$), $x=0$, $y=0$, and $z=0$.
16. $\int\int\int_W z\,dx\,dy\,dz$; W is the region bounded by the planes $x=0$, $y=0$, $z=0$, $z=1$, and the cylinder $x^2+y^2=1$, with $x\geqslant 0$, $y\geqslant 0$.
17. $\int\int\int_W x^2\cos z\,dx\,dy\,dz$; W is the region bounded by $z=0$, $z=\pi$, $y=0$, $y=1$, $x=0$, and $x+y=1$.
18. $\int_0^2\int_0^x\int_0^{x+y} dz\,dy\,dx$.
19. $\int\int\int_W (1-z^2)\,dx\,dy\,dz$; W is the pyramid with top vertex at $(0,0,1)$ and base vertices at $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$.
20. $\int\int\int_W (x^2+y^2)\,dx\,dy\,dz$; W is the same pyramid as in Exercise 19.
21. If $f(x,y,z)=F(x,y)$ for some function F—that is, if $f(x,y,z)$ does not depend on z—what is the triple integral of f over a box W?
22. Write general formulas analogous to (3) and (4) for the triple integral over a region of type III.
23. Do Example 4 by writing W as a region of type III.
24. Write out the property for triple integrals corresponding to property 2 of double integrals (Section 17.1, p. 841).
25. Show that the formula using triple integrals for the volume under the graph of a function $f(x,y)$, on an elementary region D in the plane, reduces to the double integral of f over D.
26. (a) Sketch the region for the integral
$$\int_0^1\int_0^x\int_0^y f(x,y,z)\,dz\,dy\,dx.$$
(b) Write the integral with the integration order $dx\,dy\,dz$.
27. (a) Show that the triple integral of a product over a box is the product of three ordinary integrals; that is, if $D=[a,b]\times[c,d]\times[p,q]$, then
$$\int\int\int_D f(x)g(y)h(z)\,dx\,dy\,dz$$
$$=\int_a^b f(x)\,dx\int_c^d g(y)\,dy\int_p^q h(z)\,dz.$$
(b) Use the result of part (a) to do Example 2.

17.5 Integrals in Polar, Cylindrical, and Spherical Coordinates

Problems with symmetry are often simplified by using coordinates that respect that symmetry.

We deal first with polar coordinates. Recall that a double integral
$$\int\int_D f(x,y)\,dx\,dy$$
may be thought of as a "sum" of the values of f over infinitesimal rectangles with area $(dx)\cdot(dy)$. As in Section 10.5, however, we can also describe a region using polar coordinates and can use infinitesimal regions appropriate to those coordinates. The area of such a region is $r\,dr\,d\theta$, as is evident from Fig. 17.5.1. If $u=f(x,y)$, then u may be expressed in terms of r and θ by the formula $u=f(r\cos\theta, r\sin\theta)$.

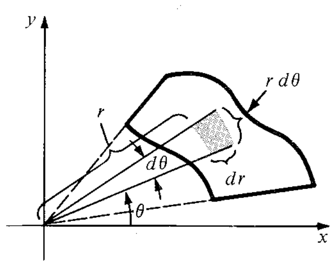


Figure 17.5.1. The area of the infinitesimal shaded region is $r\,dr\,d\theta$.

The preceding argument using infinitesimals suggests the formula in the following box (the rigorous proof is omitted).

Double Integrals in Polar Coordinates

$$\iint_D f(x, y)\,dx\,dy = \iint_{D'} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta, \tag{1}$$

where D' is the region corresponding to D in the variables r and θ.

Example 1 Evaluate $\iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy$, where D_a is the disk $x^2 + y^2 \leqslant a^2$.

Solution The presence of $r^2 = x^2 + y^2$ in the integrand and the symmetry of the disk suggest a change to polar coordinates. The disk is described by $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant 2\pi$, so by formula (1), we get

$$\iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^a e^{-r^2} r\,dr\,d\theta = \int_0^{2\pi}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^a d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi}(e^{-a^2} - 1)\,d\theta = \pi(1 - e^{-a^2}).$$

There is no direct way to evaluate this integral in xy coordinates! ▲

There is a remarkable application of the result of Example 1 to single-variable calculus: we will evaluate the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$, which is of basic importance in probability theory and quantum mechanics. There is no known way to evaluate this integral directly using only single-variable calculus. If we bring in two-variable calculus, however, the solution is surprisingly simple. Letting a go to ∞ in the formula $\iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy = \pi(1 - e^{-a^2})$, we find that the limit $L = \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy$ exists and equals π. By analogy with the definition of improper integrals on the line, we may consider L as the (improper) integral of $e^{-(x^2+y^2)}$ over the entire plane, since the disks D_a grow to fill the whole plane as $a \to \infty$. The rectangles $R_a = [-a, a] \times [-a, a]$ grow to fill the whole plane, too, so we must have

$$\lim_{a\to\infty}\iint_{R_a} e^{-(x^2+y^2)}\,dx\,dy = \pi$$

as well[3]; but

[3] The technical details of the proof that

$$\lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy = \lim_{a\to\infty}\iint_{R_a} e^{-(x^2+y^2)}\,dx\,dy$$

go as follows. We have already shown that

$$\lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy$$

exists. Thus it suffices to show that

$$\lim_{a\to\infty}\left(\iint_{R_a} e^{-(x^2+y^2)}\,dx\,dy - \iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy\right)$$

equals zero. The limit equals

$$\lim_{a\to\infty}\iint_{C_a} e^{-(x^2+y^2)}\,dx\,dy,$$

where C_a is the region between R_a and D_a (see Fig. 17.5.2). In the region C_a, $\sqrt{x^2 + y^2} \geqslant a$ (the radius of D_a), so $e^{-(x^2+y^2)} \leqslant e^{-a^2}$. Thus

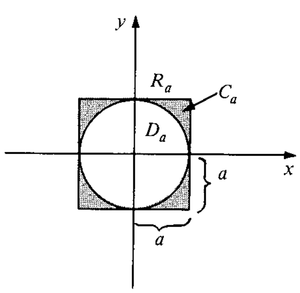


Figure 17.5.2. C_a is the shaded region between R_a and D_a.

$$\iint_{R_a} e^{-(x^2+y^2)}\,dx\,dy = \int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\,dx\,dy = \int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\,dy$$
$$= \left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right) = I_a^2,$$

where $I_a = \int_{-a}^{a} e^{-x^2}\,dx$ (see Exercise 27, Section 17.4). Thus,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \lim_{a\to\infty} I_a = \sqrt{\lim_{a\to\infty} I_a^2} = \sqrt{\lim_{a\to\infty} \iint_{R_a} e^{-(x^2+y^2)}\,dx\,dy}$$
$$= \sqrt{\pi}\,.$$

The Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}\,.$$

Example 2 Find $\int_{-\infty}^{\infty} e^{-2x^2}\,dx$.

Solution We will use the change of variables $y = \sqrt{2}\,x$ to reduce the problem to the Gaussian integral just computed.

$$\int_{-\infty}^{\infty} e^{-2x^2}\,dx = \lim_{a\to\infty}\int_{-a}^{a} e^{-2x^2}\,dx = \lim_{a\to\infty}\int_{-\sqrt{2}a}^{\sqrt{2}a} e^{-y^2}\,\frac{dy}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-y^2}\,dy = \frac{1}{\sqrt{2}}\sqrt{\pi} = \sqrt{\frac{\pi}{2}}\,. \blacktriangle$$

Example 3 Evaluate $\iint_D \ln(x^2+y^2)\,dx\,dy$, where D is the region in the first quadrant lying between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution In polar coordinates, D is described by the set of points (r,θ) such that $1 \leqslant r \leqslant 2$, $0 \leqslant \theta \leqslant \pi/2$. Hence

$$\iint_D \ln(x^2+y^2)\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=1}^{2} \ln(r^2)\,r\,dr\,d\theta$$

$$0 \leqslant \iint_{C_a} e^{-(x^2+y^2)}\,dx\,dy \leqslant \iint_{C_a} e^{-a^2}\,dx\,dy$$
$$= e^{-a^2}\,\text{area}\,(C_a) = e^{-a^2}(4a^2 - \pi a^2) = (4-\pi)a^2e^{-a^2}.$$

Thus it is enough to show that $\lim_{a\to\infty} a^2e^{-a^2} = 0$. But, by l'Hôpital's rule (see Section 11.2),

$$\lim_{a\to\infty} a^2e^{-a^2} = \lim_{a\to\infty}\left(\frac{a^2}{e^{a^2}}\right) = \lim_{a\to\infty}\left(\frac{2a}{2ae^{a^2}}\right) = \lim_{a\to\infty}\left(\frac{1}{e^{a^2}}\right) = 0,$$

as required.

$$= \int_0^{\pi/2}\int_1^2 2(\ln r)\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{r^2}{2}(2\ln r - 1)\Big|_{r=1}^{2}\right)d\theta \qquad \text{(integration by parts)}$$

$$= \int_0^{\pi/2}\left(4\ln 2 - \frac{3}{2}\right)d\theta = \frac{\pi}{2}\left(4\ln 2 - \frac{3}{2}\right). \blacktriangle$$

We will now evaluate triple integrals in cylindrical and spherical coordinates. At this point you should review the basic features of these coordinates as discussed in Section 14.5.

Cylindrical coordinates consist of polar coordinates in the xy plane, together with the z coordinate. Therefore the infinitesimal "volume element" has volume $r\,dr\,d\theta\,dz$. See Fig. 17.5.3.

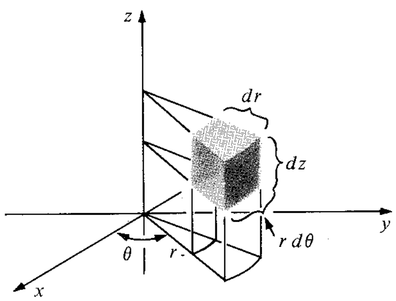


Figure 17.5.3. The infinitesimal shaded region has volume $r\,dr\,d\theta\,dz$.

As in the case of polar coordinates, this leads us to a formula for multiple integrals, presented in the next box.

Triple Integrals in Cylindrical Coordinates

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W'} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz \qquad (2)$$

where W' is the region in r,θ,z coordinates corresponding to W.

Example 4 Evaluate $\iiint_W (z^2x^2 + z^2y^2)\,dx\,dy\,dz$, where W is the cylindrical region determined by $x^2+y^2 \leqslant 1$, $-1 \leqslant z \leqslant 1$.

Solution By Formula (2) we have

$$\iiint_W (z^2x^2+z^2y^2)\,dx\,dy\,dz = \int_{-1}^{1}\int_0^{2\pi}\int_0^1 (z^2r^2)\,r\,dr\,d\theta\,dz$$

$$= \int_{-1}^{1}\int_0^{2\pi} z^2\frac{r^4}{4}\Big|_{r=0}^{1}\,d\theta\,dz$$

$$= \int_{-1}^{1}\frac{2\pi}{4}z^2\,dz = \frac{\pi}{3}. \blacktriangle$$

Finally, we turn to spherical coordinates. The volume in space corresponding to infinitesimal changes $d\rho$, $d\theta$, and $d\phi$ is shown in Fig. 17.5.4. The sides of this "box" have lengths $d\rho$, $r\,d\theta(=\rho\sin\phi\,d\theta)$, and $\rho\,d\phi$ as shown. Therefore its volume is $\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$. Hence we get the following:

Triple Integrals in Spherical Coordinates

$$\iiint_W f(x, y, z)\,dx\,dy\,dz$$
$$= \iiint_{W^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi, \qquad (3)$$

where W^* is the region in ρ, θ, ϕ space corresponding to W; i.e., the limits on ρ, θ, ϕ are chosen so that the region in xyz coordinates is W.

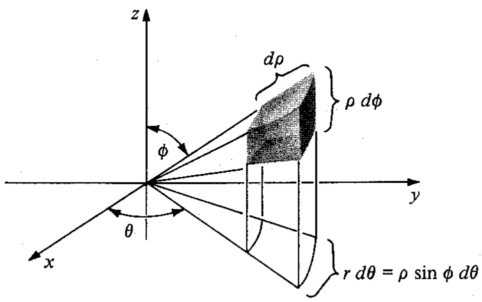


Figure 17.5.4. The infinitesimal shaded region has volume $\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$.

Example 5 Find the volume of the ball $x^2 + y^2 + z^2 \leqslant R^2$ by using spherical coordinates.

Solution The ball is described in spherical coordinates by $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \phi \leqslant \pi$, and $0 \leqslant \rho \leqslant R$. Therefore, by formula (3),

$$\iiint_W dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^R \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{R^3}{3}\int_0^{\pi}\int_0^{2\pi}\sin\phi\,d\theta\,d\phi$$
$$= \frac{2\pi R^3}{3}\int_0^{\pi}\sin\phi\,d\phi = \frac{2\pi R^3}{3}\left\{-\left[\cos(\pi) - \cos(0)\right]\right\}$$
$$= \frac{4\pi R^3}{3},$$

which is the familiar formula for the volume of a ball. Compare the effort involved with Example 4, Section 17.4. ▲

Example 6 Evaluate $\iiint_W \exp[(x^2 + y^2 + z^2)^{3/2}]\,dx\,dy\,dz$, where W is the unit ball; i.e., the set of (x, y, z) satisfying $x^2 + y^2 + z^2 \leqslant 1$.

Solution In spherical coordinates, W is described by

$$0 \leqslant \rho \leqslant 1, \qquad 0 \leqslant \phi \leqslant \pi, \qquad 0 \leqslant \theta \leqslant 2\pi.$$

Hence

$$\iiint_W \exp\left[(x^2 + y^2 + z^2)^{3/2}\right] dx\,dy\,dz$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \exp(\rho^3)\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\left(\exp(\rho^3)|_0^1\right)\sin\phi\,d\phi\,d\theta$$
$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}(e-1)\sin\phi\,d\phi\,d\theta = \frac{1}{3}(e-1)\int_0^{2\pi}\left((-\cos\phi)|_{\phi=0}^{\pi}\right)d\theta$$
$$= \frac{1}{3}(e-1)\int_0^{2\pi}2\,d\theta = \frac{2}{3}(e-1)(2\pi - 0) = \frac{4\pi}{3}(e-1). \;▲$$

It is important to spend a few moments of reflection with each integral to decide whether cylindrical, spherical, or rectangular coordinates are most useful; usually the symmetry of the problem provides the needed clue.

Example 7 Find the volumes of the following regions:

(a) The solid bounded by the circular cylinder $r = 2a\cos\theta$, the cone $z = r$, and the plane $z = 0$.
(b) The solid bounded by the cone $z = \sqrt{x^2+y^2}$ and the paraboloid of revolution $z = x^2 + y^2$.
(c) The region bounded by $y = x^2$, $y = x + 2$, $4z = x^2 + y^2$, and $z = x + 3$.

Solution The formula for the volume of a region is $\iiint_W dx\,dy\,dz$.
(a) Since we can write $r = 2a\cos\theta$ as $r^2 = 2ax$ or $(x-a)^2 + y^2 = a^2$, we see that the base of the solid is a circle in the xy plane centered at $(a, 0)$ with radius a. (See Fig. 17.5.5.) The xz plane is a plane of symmetry, so the total volume is twice the volume over the shaded region. In cylindrical coordinates, the total volume is

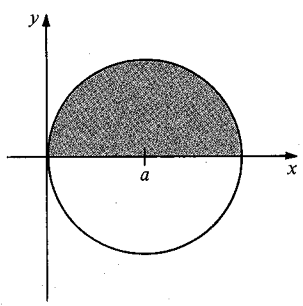


Figure 17.5.5. The base of W for Example 7(a).

$$2\int_0^{\pi/2}\int_0^{2a\cos\theta}\int_0^r r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{2a\cos\theta}\left(rz\big|_{z=0}^{r}\right)dr\,d\theta$$
$$= 2\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\,dr\,d\theta = 2\int_0^{\pi/2}\left(\frac{r^3}{3}\bigg|_{r=0}^{2a\cos\theta}\right)d\theta$$
$$= 2\int_0^{\pi/2}\frac{8a^3\cos^3\theta}{3}\,d\theta = \left(\frac{16a^3}{3}\right)\int_0^{\pi/2}(1-\sin^2\theta)\cos\theta\,d\theta.$$

Let $u = \sin\theta$ to get $(16a^3/3)\int_0^1(1-u^2)\,du = (16a^3/3)(u - u^3/3)|_0^1 = 32a^3/9$.
(b) In cylindrical coordinates, the solid is bounded by $z = r$ and $z = r^2$ (Fig. 17.5.6.) The solid is obtained by rotating the shaded area around the z axis. Thus, the volume is

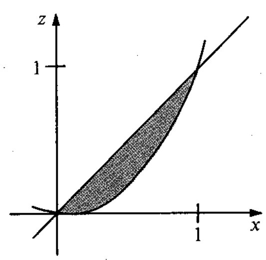


Figure 17.5.6. A cross section of W for Example 7(b).

$$\int_0^{2\pi}\int_0^1\int_{r^2}^r r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\left(rz\big|_{z=r^2}^{r}\right)dr\,d\theta = \int_0^{2\pi}\int_0^1(r^2-r^3)\,dr\,d\theta$$
$$= \int_0^{2\pi}\left(\frac{r^3}{3}-\frac{r^4}{4}\right)\bigg|_{r=0}^{1}d\theta = \frac{1}{12}\int_0^{2\pi}d\theta = \frac{\pi}{6}.$$

(c) This part does *not* require cylindrical or spherical coordinates; $y = x^2 = x + 2$ has the solutions $x = -1$ and $x = 2$, so the volume is

$$\int_{-1}^2\int_{x^2}^{x+2}\int_{(x^2+y^2)/4}^{x+3} dz\,dy\,dx = \int_{-1}^2\int_{x^2}^{x+2}\left[(x+3)-\frac{x^2+y^2}{4}\right]dy\,dx$$
$$= \int_{-1}^2\left\{\left[\left(x+3-\frac{x^2}{4}\right)y - \frac{y^3}{12}\right]\bigg|_{y=x^2}^{x+2}\right\}dx$$
$$= \int_{-1}^2\left(\frac{16}{3}+4x-3x^2-\frac{4x^3}{3}+\frac{x^4}{4}+\frac{x^6}{12}\right)dx$$
$$= \left(\frac{16x}{3}+2x^2-x^3-\frac{x^4}{3}+\frac{x^5}{20}+\frac{x^7}{84}\right)\bigg|_{-1}^{2} = \frac{783}{70}.$$

Exercises for Section 17.5

1. Evaluate $\iint_D (x^2+y^2)^{3/2}\,dx\,dy$, where D is the disk $x^2+y^2 \leqslant 4$.
2. Evaluate $\iint_D (x^2+y^2)^{5/2}\,dx\,dy$; D is the disk $x^2+y^2 \leqslant 1$.
3. Evaluate $\int_{-\infty}^{\infty} e^{-10x^2}\,dx$.
4. Evaluate $\int_{-\infty}^{\infty} 3e^{-8x^2}\,dx$.
5. Integrate x^2+y^2 over the disk of radius 4 centered at the origin.
6. Find $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x^2+y^2)\,dy\,dx$ by converting to polar coordinates.
7. Integrate $ze^{x^2+y^2}$ over the cylinder $x^2+y^2 \leqslant 4$, $2 \leqslant z \leqslant 3$.
8. Integrate $x^2+y^2+z^2$ over the cylinder given by $x^2+z^2 \leqslant 2$, $-2 \leqslant y \leqslant 3$.
9. Evaluate
$$\iiint_W \frac{dx\,dy\,dz}{\sqrt{1+x^2+y^2+z^2}},$$
where W is the ball $x^2+y^2+z^2 \leqslant 1$.
10. Evaluate $\iiint_W (x^2+y^2+z^2)^{5/2}\,dx\,dy\,dz$; W is the ball $x^2+y^2+z^2 \leqslant 1$.
11. Evaluate
$$\iiint_S \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}},$$
where S is the solid bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, where $a>b>0$.
12. Integrate $\sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}$ over the region in Exercise 11.
13. Find the volume of the region bounded by the surfaces $x^2+y^2+z^2=1$ and $x^2+y^2=\frac{1}{4}$.
14. Find the volume of the region enclosed by the cones $z=\sqrt{x^2+y^2}$ and $z=1-2\sqrt{x^2+y^2}$.
15. Find the volume inside the ellipsoid $x^2+y^2+4z^2=6$.
16. Find the volume of the intersection of the ellipsoid $x^2+2(y^2+z^2) \leqslant 10$ and the cylinder $y^2+z^2 \leqslant 1$.
17. Find the *normalizing constant* c, depending on σ, such that $\int_{-\infty}^{\infty} ce^{-x^2/\sigma}\,dx=1$.
18. Integrate $(x^2+y^2)z^2$ over the part of the cylinder $x^2+y^2 \leqslant 1$ inside the sphere $x^2+y^2+z^2=4$.

★19. The general change of variables formula in two dimensions reads
$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} h(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv,$$
where $h(u,v)=f(x(u,v),y(u,v))$ and where $|\partial(x,y)/\partial(u,v)|$ is the absolute value of the determinant
$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$
Here $x(u,v)$ and $y(u,v)$ are the functions relating the variables (u,v) to the variables (x,y), and D^* is the region in the uv plane which corresponds to D.
(a) Show that this formula is plausible by using the geometric interpretation of derivatives and determinants.
(b) Show that the formula reduces to our earlier one when u and v are polar coordinates.

★20. Using the idea of Exercise 19, write down the general three-dimensional change of variables formula and show that it reduces to our earlier ones for cylindrical and spherical coordinates.

★21. By using the change of variables formula in Exercise 19 and $u=x+y$, $y=uv$, show that
$$\int_0^1\int_0^{1-x} e^{y/(x+y)}\,dy\,dx = \frac{e-1}{2}.$$
Also graph the region in the xy plane and the uv plane.

★22. Let D be the region bounded by $x+y=1$, $x=0$, $y=0$. Use the result of Exercise 19 to show that
$$\iint_D \cos\left(\frac{x-y}{x+y}\right)dx\,dy = \frac{\sin 1}{2},$$
and graph D on an xy plane and a uv plane, with $u=x-y$ and $v=x+y$.

17.6 Applications of Triple Integrals

The calculation of mass and center of mass of a region in space involves triple integrals.

Some of the applications in Section 17.3 carry over directly from double to triple integrals. We can compute the volume, mass, and center of mass of a region with variable density $\rho(x, y, z)$ by the formulas in the following box.

Volume, Mass, Center of Mass, and Average Value

$$\text{Volume} = \iiint_W dx\,dy\,dz,$$
$$\text{Mass} = \iiint_W \rho(x, y, z)\,dx\,dy\,dz. \tag{1}$$

Center of mass $= (\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{\iiint_W x\rho(x, y, z)\,dx\,dy\,dz}{\text{mass}},$$
$$\bar{y} = \frac{\iiint_W y\rho(x, y, z)\,dx\,dy\,dz}{\text{mass}}, \tag{2}$$
$$\bar{z} = \frac{\iiint_W z\rho(x, y, z)\,dx\,dy\,dz}{\text{mass}}.$$

The *average value* of a function f on a region W is defined by

$$\frac{\iiint_W f(x, y, z)\,dx\,dy\,dz}{\iiint_W dx\,dy\,dz}. \tag{3}$$

Example 1 The cube $[1,2] \times [1,2] \times [1,2]$ has mass density $\rho(x, y, z) = (1 + x)e^z y$. Find the mass of the box.

Solution The mass of the box is

$$\begin{aligned}
&\int_1^2\int_1^2\int_1^2 (1 + x)e^z y\,dx\,dy\,dz \\
&= \int_1^2\int_1^2\left[\left(x + \frac{x^2}{2}\right)e^z y\right]_{x=1}^{x=2} dy\,dz = \int_1^2\int_1^2 \frac{5}{2}e^z y\,dy\,dz \\
&= \int_1^2 \frac{15}{4}e^z\,dz = \left[\frac{15}{4}e^z\right]_{z=1}^{z=2} \\
&= \frac{15}{4}(e^2 - e). \blacktriangle
\end{aligned}$$

Example 2 Find the center of mass of the hemispherical region W defined by the inequalities $x^2 + y^2 + z^2 \leqslant 1$, $z \geqslant 0$. (Assume that the density is constant.)

Solution By symmetry, the center of mass must lie on the z axis, so $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we must compute, by formula (2), $I = \iiint_W z\,dx\,dy\,dz$. The hemisphere is of types I, II, and III; we will consider it to be of type III. Then the integral I becomes

$$I = \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} z\,dx\,dy\,dz.$$

Since z is a constant for the x and y integrations, we can bring it out to obtain

$$I = \int_0^1 z\left(\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} dx\,dy\right) dz.$$

Instead of calculating the inner two integrals explicitly, we observe that they are simply the double integral $\iint_D dx\,dy$ over the disk $x^2 + y^2 \leqslant 1 - z^2$, considered as a type 2 region. The area of this disk is $\pi(1 - z^2)$, so

$$I = \pi\int_0^1 z(1 - z^2)\,dz = \pi\int_0^1 (z - z^3)\,dz = \pi\left[\frac{z^2}{2} - \frac{z^4}{4}\right]_0^1 = \frac{\pi}{4}.$$

The volume of the hemisphere is $(2/3)\pi$, so $\bar{z} = (\pi/4)/[(2/3)\pi] = 3/8$. ▲

Example 3 The temperature at points in the cube $W = [-1,1] \times [-1,1] \times [-1,1]$ is proportional to the square of the distance from the origin.

(a) What is the average temperature?

(b) At which points of the cube is the temperature equal to the average temperature?

Solution (a) Let c be the constant of proportionality. Then $T = c(x^2 + y^2 + z^2)$ and the average temperature is $\bar{T} = \frac{1}{8}\iiint_W T\,dx\,dy\,dz$, since the volume of the cube is 8. Thus

$$\bar{T} = \frac{c}{8}\int_{-1}^1\int_{-1}^1\int_{-1}^1 (x^2 + y^2 + z^2)\,dx\,dy\,dz.$$

The triple integral is the sum of the integrals of x^2, y^2, and z^2. Since x, y, and z enter symmetrically into the description of the cube, the three integrals will be equal, so

$$\bar{T} = \frac{3c}{8}\int_{-1}^1\int_{-1}^1\int_{-1}^1 z^2\,dx\,dy\,dz = \frac{3c}{8}\int_{-1}^1 z^2\left(\int_{-1}^1\int_{-1}^1 dx\,dy\right) dz.$$

The inner integral is equal to the area of the square $[-1,1] \times [-1,1]$. The area of that square is 4, so

$$\bar{T} = \frac{3c}{8}\int_{-1}^1 4z^2\,dz = \frac{3c}{2}\left(\frac{z^3}{3}\right)\Bigg|_{-1}^1 = c.$$

(b) The temperature is equal to the average temperature when $c(x^2 + y^2 + z^2) = c$; that is, on the sphere $x^2 + y^2 + z^2 = 1$, which is inscribed in the cube W. ▲

Example 4 The *moment of inertia* about the x axis of a solid S with uniform density ρ is defined by

$$I_x = \iiint_S \rho(y^2 + z^2)\,dx\,dy\,dz.$$

Similarly,

$$I_y = \iiint_S \rho(x^2 + z^2)\,dx\,dy\,dz, \qquad I_z = \iiint_S \rho(x^2 + y^2)\,dx\,dy\,dz.$$

For the following solid, compute I_z; assume that the density is a constant: The solid above the xy plane, bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$.

Solution The paraboloid and cylinder intersect at the plane $z = a^2$. Using cylindrical coordinates, we find

$$I_z = \int_0^a \int_0^{2\pi} \int_0^{r^2} \rho r^2 \cdot r\,dz\,d\theta\,dr = \rho \int_0^a \int_0^{2\pi} \int_0^{r^2} r^3\,dz\,d\theta\,dr = \frac{\pi\rho a^6}{3}. \blacktriangle$$

per unit mass

An interesting physical application of triple integration is the determination of the gravitational fields of solid objects. Example 3, Section 16.2, showed that the gravitational force field $\mathbf{F}(x, y, z)$ is the negative of the gradient of a function $V(x, y, z)$ called the *gravitational potential.* If there is a point mass m at (x, y, z), then the gravitational potential at (x_1, y_1, z_1) due to this mass is $-Gm[(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{-1/2}$, where G is the universal gravitational constant.

If our attracting object is an extended domain W with density $\rho(x, y, z)$, we may think of it as made of infinitesimal box-shaped regions with masses $dm = \rho(x, y, z)\,dx\,dy\,dz$ located at points (x, y, z). The total gravitational potential for W is then obtained by "summing" the potentials from the infinitesimal masses—that is, as a triple integral (see Fig. 17.6.1):

$$V(x_1, y_1, z_1) = -G \iiint_W \frac{\rho(x, y, z)\,dx\,dy\,dz}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}}. \tag{4}$$

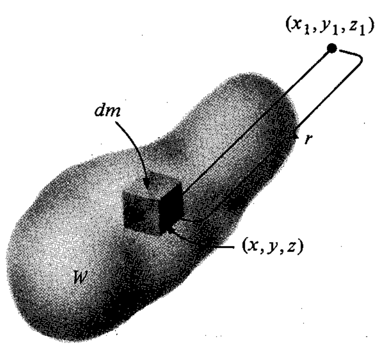


Figure 17.6.1. The gravitational potential at (x_1, y_1, z_1) arising from the mass $dm = \rho(x, y, z)\,dx\,dy\,dz$ at (x, y, z) is $-[G\rho(x, y, z)\,dx\,dy\,dz]/r$.

The evaluation of the integral for the gravitational potential is usually quite difficult. The few examples which can be carried out completely require the use of cylindrical or spherical coordinate systems.

Historical Note Newton withheld publication of his gravitational theories for quite some time, until he could prove that a spherical planet has the same gravitational field that it would have if its mass were all concentrated at the planet's center. Using multiple integrals and spherical coordinates, we shall solve Newton's problem below; Newton's published solution used only euclidean geometry.

Example 5 Let W be a region of constant density and total mass M. Show that the gravitational potential for a unit mass is given by

$$V(x_1, y_1, z_1) = -\overline{\left(\frac{1}{r}\right)} GM,$$

where $\overline{(1/r)}$ is the average over W of

$$f(x, y, z) = \frac{1}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}}.$$

Solution According to formula (4),

$$V(x_1, y_1, z_1) = -G\iiint_W \frac{\rho(x, y, z)\, dx\, dy\, dz}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}}$$

$$= -G\rho\iiint_W \frac{dx\, dy\, dz}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}}$$

$$= -G[\rho \text{ volume } (W)] \frac{\displaystyle\iiint_W \frac{dx\, dy\, dz}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}}}{\text{volume } (W)}$$

$$= -GM\overline{\left(\frac{1}{r}\right)},$$

as required. ▲

Let us now use formula (4) and spherical coordinates to find the gravitational potential $V(x_1, y_1, z_1)$ for the region W between the concentric spheres $\rho = \rho_1$ and $\rho = \rho_2$, assuming the density is constant. Before evaluating the integral in formula (4), we make some observations which will simplify the computation. Since G and the density are constants, we may ignore them at first. Since the attracting body W is symmetric with respect to all rotations about the origin, the potential $V(x_1, y_1, z_1)$ must itself be symmetric—thus $V(x_1, y_1, z_1)$ depends only on the distance $R = \sqrt{x_1^2 + y_1^2 + z_1^2}$ from the origin. Our computation will be simplest if we look at the point $(0, 0, R)$ on the z axis (see Fig. 17.6.2). Thus our integral is

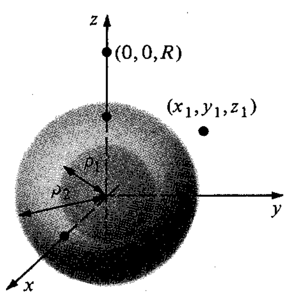


Figure 17.6.2. The gravitational potential at (x_1, y_1, z_1) is the same as at $(0, 0, R)$, where $R = \sqrt{x_1^2 + y_1^2 + z_1^2}$.

$$V(0, 0, R) = -\iiint_W \frac{dx\, dy\, dz}{\sqrt{x^2 + y^2 + (z - R)^2}}.$$

In spherical coordinates, W is described by the inequalities $\rho_1 \leqslant \rho \leqslant \rho_2$, $0 \leqslant \theta \leqslant 2\pi$, and $0 \leqslant \phi \leqslant \pi$, so by formula (3) in Section 17.5,

$$V(0, 0, R) = -\int_{\rho_1}^{\rho_2}\int_0^{\pi}\int_0^{2\pi} \frac{\rho^2 \sin\phi\, d\theta\, d\phi\, d\rho}{\sqrt{\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) + (\rho\cos\phi - R)^2}}.$$

Replacing $\cos^2\theta + \sin^2\theta$ by 1, so that the integrand no longer involves θ, we may integrate over θ to get

$$V(0,0,R) = 2\pi \int_{\rho_1}^{\rho_2} \int_0^{\pi} \frac{\rho^2 \sin\phi \, d\phi \, d\rho}{\sqrt{\rho^2 \sin^2\phi (\rho\cos\phi - R)^2}}$$

$$= 2\pi \int_{\rho_1}^{\rho_2} \rho^2 \left[\int_0^{\pi} \frac{\sin\phi \, d\phi}{\sqrt{\rho^2 - 2R\rho\cos\phi + R^2}} \right] d\rho.$$

The inner integral is easily evaluated by the substitution $u = -2R\rho\cos\phi$: it becomes

$$\frac{1}{2R\rho} \int_{-2R\rho}^{2R\rho} (\rho^2 + u + R^2)^{-1/2} du = \frac{2}{2R\rho} (\rho^2 + u + R^2)^{1/2} \Big|_{-2R\rho}^{2R\rho}$$

$$= \frac{1}{R\rho} \left[(\rho^2 + 2R\rho + R^2)^{1/2} - (\rho^2 - 2R\rho + R^2)^{1/2} \right]$$

$$= \frac{1}{R\rho} \left\{ \left[(\rho + R)^2 \right]^{1/2} - \left[(\rho - R)^2 \right]^{1/2} \right\}$$

$$= \frac{1}{R\rho} (\rho + R - |\rho - R|).$$

The expression $\rho + R$ is always positive, but $\rho - R$ may not be, so we must keep the absolute value sign. (Here we have used the formula $\sqrt{x^2} = |x|$.) Substituting into the formula for V, we get

$$V(0,0,R) = 2\pi \int_{\rho_1}^{\rho_2} \frac{\rho^2}{R\rho} (\rho + R - |\rho - R|) \, d\rho$$

$$= \frac{2\pi}{R} \int_{\rho_1}^{\rho_2} \rho(\rho + R - |\rho - R|) \, d\rho.$$

We will now consider two possibilities for R, corresponding to the gravitational potential for objects outside and inside the hollow ball W.

If $R \geqslant \rho_2$ (that is, (x_1, y_1, z_1) is outside W), then $|\rho - R| = R - \rho$ for all ρ in the interval $[\rho_1, \rho_2]$, so

$$V(0,0,R) = \frac{2\pi}{R} \int_{\rho_1}^{\rho_2} \rho[\rho + R - (R - \rho)] \, d\rho$$

$$= \frac{4\pi}{R} \int_{\rho_1}^{\rho_2} \rho^2 \, d\rho = \frac{1}{R} \frac{4\pi}{3} (\rho_2^3 - \rho_1^3).$$

The factor $(4\pi/3)(\rho_2^3 - \rho_1^3)$ is just the volume of W. Putting back the constants G and the mass density we find that *the gravitational potential is GM/R, where M is the mass of W. Thus V is just as it would be if all the mass of W were concentrated at the central point* (see Example 3, Section 16.2.)

If $R \leqslant \rho_1$ (that is, (x_1, y_1, z_1) is inside the hole), then $|\rho - R| = \rho - R$ for ρ in $[\rho_1, \rho_2]$, and

$$V(0,0,R) = \frac{2\pi}{R} \int_{\rho_1}^{\rho_2} \rho[\rho + R - (\rho - R)] \, d\rho = 4\pi \int_{\rho_1}^{\rho_2} \rho \, d\rho = 2\pi(\rho_2^2 - \rho_1^2).$$

The result is independent of R, so the potential V is constant inside the hole. Since the gravitational force is minus the gradient of V, we conclude that *there is no gravitational force inside a uniform hollow planet*!

We leave it to you (Review Exercise 47) to compute $V(0,0,R)$ for the case $\rho_1 < R < \rho_2$.

A similar argument shows that the gravitational potential outside any spherically symmetric body of mass M (even if the density is variable) is

$V = -GM/R$, where R is the distance to its center (which is its center of mass) (Exercise 17).

Example 6 Find the gravitational potential of a spherical star with a mass $M = 3.02 \times 10^{30}$ kilograms at a distance of 2.25×10^{11} meters from its center ($G = 6.67 \times 10^{-11}$ Newton meter²/kilogram²).

Solution The potential is

$$V = \frac{-GM}{R} = \frac{-6.67 \times 10^{-11} \times 3.02 \times 10^{30}}{2.25 \times 10^{11}} = -8.95 \times 10^{8} \text{ meters}^2/\text{sec}^2. \blacktriangle$$

Exercises for Section 17.6

1. (a) Find the mass of the box $[0, \frac{1}{2}] \times [0, 1] \times [0, 2]$, assuming the density to be uniform. (b) Same Exercise as part (a), but with a mass density $\rho(x, y, z) = x^2 + 3y^2 + z + 1$.
2. Find the mass of the solid bounded by the cylinder $x^2 + z^2 = 2x$ and the cone $z^2 = x^2 + y^2$ if the density is $\rho = \sqrt{x^2 + y^2}$.

Find the center of mass of the solids in Exercises 3 and 4, assuming them to have constant density.

3. S bounded by $x + y + z = 2$, $x = 0$, $y = 0$, $z = 0$.
4. S bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 6$, $z = 0$.
5. Evaluate the integral in Example 2 by considering the hemisphere as a region of type I.
6. Find the center of mass of the cylinder $x^2 + y^2 \leqslant 1$, $1 \leqslant z \leqslant 2$ if the density is $\rho = (x^2 + y^2)z^2$.
7. Redo Example 3 for the cube

$$W = [-c, c] \times [-c, c] \times [-c, c].$$

[*Hint*: Guess the answer to part (b) first.]

8. Find the average value of $x^2 + y^2$ over the conical region $0 \leqslant z \leqslant 2$, $x^2 + y^2 \leqslant z^2$.
9. Find the average value of $\sin^2 \pi z \cos^2 \pi x$ over the cube $[0, 2] \times [0, 4] \times [0, 6]$.
10. Find the average value of e^{-z} over the ball $x^2 + y^2 + z^2 \leqslant 1$.
11. A solid with constant density is bounded above by the plane $z = a$ and below by the cone described in spherical coordinates by $\phi = k$, where k is a constant $0 < k < \pi/2$. Set up an integral for its moment of inertia about the z axis.
12. Find the moment of inertia around the y axis for the ball $x^2 + y^2 + z^2 \leqslant R^2$ if the mass density is a constant ρ.
13. Find the gravitational potential of a spherical planet with mass $M = 3 \times 10^{26}$ kilograms, at a distance of 2×10^8 meters from its center.
14. Find the gravitational force exerted on a 70-kilogram object at the position in Exercise 13. (See Example 3, Section 16.2.)
15. A body W in xyz coordinates is *symmetric with respect to a plane* if for every particle on one side of the plane there is a particle of equal mass located at its mirror image through the plane.
    (a) Discuss the planes of symmetry for the body of an automobile.
    (b) Let the plane of symmetry be the xy plane, and denote by $W+$ and $W-$ the portions of W above and below the plane, respectively. By our assumption, the mass density $\rho(x, y, z)$ satisfies $\rho(x, y, -z) = \rho(x, y, z)$. Justify these steps:

$$\begin{aligned} \bar{z} \cdot \iiint_W \rho(x, y, z)\, dx\, dy\, dz &= \iiint_W z\rho(x, y, z)\, dx\, dy\, dz \\ &= \iiint_{W+} z\rho(x, y, z)\, dx\, dy\, dz \\ &\quad + \iiint_{W-} z\rho(x, y, z)\, dx\, dy\, dz \\ &= \iiint_{W+} z\rho(x, y, z)\, dx\, dy\, dz \\ &\quad + \iiint_{W+} -w\rho(u, v, -w)\, du\, dv\, dw = 0. \end{aligned}$$

    (c) Explain why part (b) proves that if a body is symmetrical with respect to a plane, then its center of mass lies in that plane.
    (d) Derive this law of mechanics: "If a body is symmetrical in two planes, then its center of mass lies on their line of intersection."

★16. If a body is composed of two or more parts whose centers of mass are known, then the center of mass of the composite body can be computed by regarding the component parts as single particles located at their respective centers of mass. Apply this consolidation principle below.
    (a) Find the center of mass of an aluminum block of constant density ρ, of base 4×6 centimeters, height 10 centimeters, with a hole drilled through. The cylinder removed is 2 centimeters in diameter and 6 centimeters long with its axis of symmetry 8 centi-

meters above the base and symmetrically placed.

(b) Repeat for a solid formed by pouring epoxy into a hemispherical form of radius 20 centimeters which contains a balloon of diameter 8 centimeters placed at the center of the circular base.

★17. Show that the gravitational potential outside of a spherically symmetric body whose density is a given function of the radius is $V(x_1, y_1, z_1) = GM/R$, where M is the mass of the body and $R = \sqrt{x_1^2 + y_1^2 + z_1^2}$ is the distance to the center of the body.

Review Exercises for Chapter 17

Evaluate the integrals in Exercises 1–10.

1. $\int_2^3\int_4^8[x^3 + \sin(x+y)]\,dx\,dy$.
2. $\int\int_D[x^3 + \sin(x+y)]\,dx\,dy$; D is the rectangle $[1,2]\times[-3,2]$.
3. $\int_1^2\int_2^3\int_3^4(x+y+z)\,dx\,dy\,dz$.
4. $\int\int\int_W[e^x + (y+z)^5]\,dx\,dy\,dz$; W is the cube $[0,1]\times[0,1]\times[0,1]$.
5. $\int\int_D(x^3 + y^2x)\,dx\,dy$; D = region under the graph of $y = x^2$ from $x = 0$ to $x = 2$.
6. $\int\int\int_W(x^2+y^2+z^2)\,dx\,dy\,dz$; W = solid hemisphere $x^2+y^2+z^2 \leqslant 1$, $z \geqslant 0$.
7. $\int\int_D\sec(x^2+y^2)\,dx\,dy$, where D is the region defined by $x^2+y^2 \leqslant 1$.
8. $\int\int\int_W(x^2+8yz)\,dx\,dy\,dz$, where W is the region bounded by the surfaces $z = x^2 - y^2$, $z = 0$, $y = \pm 1$, and $x = 0, 4$.
9. $\int_0^1\int_0^x\int_0^y xyz\,dz\,dy\,dx$.
10. $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos(x+y+z)\,dx\,dy\,dz$.

Find the volume of each of the solids in Exercises 11–20.

11. The region bounded by the five planes $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$, and the paraboloid $z = x^2 + y^2$.
12. The cone defined by $(x - \frac{1}{2}z)^2 + y^2 \leqslant 4z^2$, and $0 \leqslant z \leqslant 3$.
13. The ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 \leqslant 1$.
14. The intersection of a ball B of radius 1 with a ball of radius $\frac{1}{2}$ whose center is on the boundary of B.
15. The spherical sector $x^2+y^2+z^2 \leqslant 1$, $z \geqslant 0$, $x^2+y^2 \leqslant az^2$.
16. The region between the graphs of $f(x,y) = \cos^2(y+x)$ and $g(x,y) = -\sin^2(y+x)$ on the domain $D = [0,1]\times[0,1]$.
17. The "ice cream cone" defined by $x^2+y^2 \leqslant \frac{1}{5}z^2$, $0 \leqslant z \leqslant 5 + \sqrt{5 - x^2 - y^2}$.
18. The region below the plane $z + y = 1$ and inside the cylinder $x^2+y^2 \leqslant 1$, $0 \leqslant z \leqslant 1$.
19. The solid bounded by $x^2+y^2+z^2 = 1$ and $z^2 \geqslant x^2+y^2$.
20. The solid bounded by $x^2+y^2+z^2 = 1$ and $z \geqslant x^2+y^2$.

Sketch and find the volume under the graph of f between the planes $x = a$, $x = b$, $y = c$, and $y = d$ in Exercises 21 and 22.

21. $f(x,y) = x^2 + \sin(2y) + 1$; $a = -3$, $b = 1$, $c = 0$, $d = \pi$.
22. $f(x,y) = 10 - x^2 - y^2$; $a = -2$, $b = 2$, $c = -1$, $d = 1$.

23. Find the average value of $f(x,y)$ on $D = [a,b]\times[c,d]$ for the function and region given in Review Exercise 21.
24. Find the average value of $f(x,y)$ on $D = [a,b]\times[c,d]$ for the function and region given in Review Exercise 22.
25. The *tetrahedron* defined by $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, $x + y + z \leqslant 1$ is to be sliced into n segments of equal volume by planes parallel to the plane $x + y + z = 1$. Where should the slices be made?
26. Show that the volume obtained by cutting the rectangular cylinder $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ by the planes $z = 0$ and $z = px + qy + r$ is equal to the area of the base times the average of the heights of the four vertical edges. Assume that $px + qy + r \geqslant 0$ for all (x,y) in the rectangle $[a,b]\times[c,d]$. (See Fig. 17.R.1.)

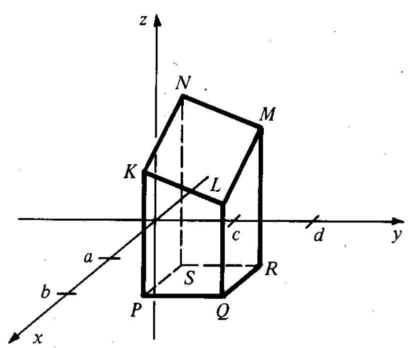


Figure 17.R.1. The volume of solid $PQRSKLMN$ is equal to the area of the base $PQRS$ times the average of the lengths of PK, QL, RM, and SN.

27. Show that the surface area of the part of the sphere $x^2 + y^2 + z^2 = 1$ lying above the rectangle $[-a,a] \times [-a,a]$ in the xy plane is

$$A = 2\int_{-a}^{a} \sin^{-1}\left(\frac{a}{\sqrt{1-x^2}}\right) dx \qquad (2a^2 < 1).$$

28. The sphere $x^2 + y^2 + z^2 = 1$ is to be cut into three pieces of equal surface area by two parallel planes. How should this be done?
29. Use cylindrical coordinates to find the center of mass of the "dish":

$$x^2 + y^2 \leqslant 1, \quad x^2 + y^2 + (z-2)^2 \leqslant 25/4, \quad z \leqslant 0.$$

30. Use cylindrical coordinates to find the center of mass of the region

$$y^2 + z^2 \leqslant 1/4, \quad (x-1)^2 + y^2 + z^2 \leqslant 1, \quad x \leqslant 1.$$

Using polar coordinates, find the surface area of the graph of each of the functions in Exercises 31–34 over the unit disk $x^2 + y^2 \leqslant 1$ (express your answer as an integral if necessary).

31. xy
32. $x^2 + y^2$
33. $x^2 - y^2$
34. $(x^2 + y^2)^2$

Evaluate the integrals in Exercises 35–38.

35. $\int_0^{\pi}\int_0^{\pi/2}\int_0^{\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$
36. $\int_0^1\int_0^{\pi/2}\int_0^{\cos\theta} rz \, dr \, d\theta \, dz.$
37. $\int\int_D dx\,dy/(x^2+y^2)$, where D is the region in the first quadrant bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$.
38. $\int\int_{[0,1]\times[0,1]} e^{yx}\,dy\,dx$. (Use a power series to evaluate to within 0.001.)
39. As is well known, the density of a typical planet is not constant throughout the planet. Assume that planet I.K.U. has a radius of 5×10^8 centimeters and a mass density (in grams per cubic centimeter)

$$\rho(x,y,z) = \begin{cases} \dfrac{3\times 10^4}{r}, & r \geqslant 10^4 \text{ centimeters,} \\ 3, & r \leqslant 10^4 \text{ centimeters,} \end{cases}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Find a formula for the gravitational potential outside I.K.U.

40. Derive the following three laws (see Exercise 15, Section 17.6).
    (a) If a body is symmetrical about an axis, then its center of mass lies on that axis.
    (b) If a body is symmetrical in three planes with a common point, then that point is its center of mass.
    (c) If a body has spherical symmetry about a point (that is, if the density depends only on the distance from that point), then that point is its center of mass.
41. The *flexural rigidity EI* of a uniform beam is the product of its Young's modulus of elasticity E and the moment of inertia I of the cross section of the beam at x with respect to a horizontal line l passing through the center of gravity of this cross section. Here

$$I = \int\int_R [D(x,y)]^2 \, dx\,dy,$$

where $D(x,y)$ = the distance from (x,y) to l, R = the cross section of the beam being considered.
    (a) Assume the cross section R is the rectangle $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 2$ and l is the x axis. Find I.
    (b) Assume the cross section R is a circle of radius 4, and l is the x axis. Find I, using polar coordinates.
42. Justify the formula

$$\int\int_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\right) dx$$

for an integral over a region of type 1 by using the slice method.

43. Find $\int_{-\infty}^{\infty} e^{-5x^2}\,dx.$
44. Find $\int_{-\infty}^{\infty} (5 + x^2)e^{-2x^2}\,dx.$
45. (a) Interpret the result in Exercise 26 as a fact about the average value of a linear function on a rectangle.
    ★(b) What is the average value of a linear function on a parallelogram?

★46. If the world were two-dimensional,[4] the laws of physics would predict that the gravitational potential of a mass point is proportional to the logarithm of the distance from the point. Using polar coordinates, write an integral giving the gravitational potential of a disk of constant density.

★47. Find the gravitational potential in the situation of Fig. 17.6.2 when $\rho_1 < R < \rho_2$. [*Hint*: Break the integral over ρ into two parts.]

★48. There is an interesting application of double integrals to the problem of differentiation under the integral sign.[5] Study the following theorem:

Let $D = [a,b] \times [c,d]$ and let $g(x,y)$ be continuous, with $g_x(x,y)$ continuous on D. Then

$$\frac{d}{dx}\int_c^d g(x,y)\,dy = \int_c^d g_x(x,y)\,dy, \qquad a < x < b.$$

[4] See *Flatland* by Edwin A. Abbot (Barnes and Noble, 1963) for an amusing (but sexist) description of such a world.

[5] See "Fubini implies Leibniz implies $F_{yx} = F_{xy}$" by R. T. Seeley, American Mathematical Monthly **68**(1961), 56–57.

The idea of the proof is this: By the fundamental theorem of calculus,

$$\int_c^d g(x, y)\,dy = \int_c^d \left[\int_a^x g_x(s, y)\,ds + g(a, y) \right] dy.$$

Interchanging the order of integration, we get

$$\int_a^x \left[\int_c^d g_x(s, y)\,dy \right] ds + \int_c^d g(a, y)\,dy.$$

We can show that $\int_c^d g_x(s, y)\,dy$ is a continuous function of s, so we can use the fundamental theorem of calculus (alternative version) to get

$$\frac{d}{dx}\int_c^d g(x, y)\,dy = \frac{d}{dx}\int_a^x \left[\int_c^d g_x(s, y)\,dy \right] ds + \frac{d}{dx}\left[\int_c^d g(a, y)\,dy \right] = \int_c^d g_x(x, y)\,dy + 0$$

as asserted.

(a) Verify by direct integration that

$$\frac{d}{dx}\int_0^{\pi/2} \sin(xy)\,dy = \int_0^{\pi/2} \frac{\partial}{\partial x}\sin(xy)\,dy.$$

(b) Show that $F(k) = \int_0^{\pi/2} dx/\sqrt{1 - k\cos^2 x}$, for $0 \leqslant k < 1$, is an increasing function of k. How fast is it increasing at $k = 0$?

★49. Use the discussion of the theorem in Exercise 48 to show that interchanging the order of integration allows one to prove that the mixed partials of a function are equal.

★50.[6] Show that each of the following functions has the curious property that the volume under its graph equals the surface area of its graph on any region:

(a) $f(x, y) = 1$,

(b) $f(x, y) = \cosh(x\cos\alpha + y\sin\alpha + c)$, ($\alpha, c$ constants) and

(c) $f(x, y) = \cosh(\sqrt{x^2 + y^2} + c)$ (c constant). (Compare Review Exercise 85, Chapter 8).

★51. Suppose that $f(x, y) = \cosh[u(x, y)]$ has the property considered in Review Exercise 50. Find a partial differential equation satisfied by $u(x, y)$ if u is not identically zero (i.e., find a relation involving u, u_x and u_y).

[6] Suggested by Chris Fisher.